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ABSTRACT

This twenty-second unit in the SMSG secondary school mathematics series is the teacher's commentary for Unit 21. For each of the chapters in Unit 21, a time allotment is suggested, the goals for that chapter are discussed, the mathematics is explained, some teaching suggestions are given, answers to exercises are provided, and sample test questions are included. In the appendices, mathematical induction is briefly discussed, then solutions to problems given in the appendices of Unit 21 are provided. (DT)

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School Mathematics Study Group

Elementary Functions

Unit 22

Elementary Functions

Teacher's Commentary

REVISED EDITION

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the Panel on Sample Textbooks
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Commentary for Teachers

ELEMENTARY FUNCTIONS

Introduction.

The text, Elementary Functions, and an accompanying commentary were produced by a team of high school and college mathematics teachers during the summer of 1959 for the School Mathematics Study Group. It was designed for use in a one-semester course in the 12th grade. Both text and commentary have been completely revised during the summer of 1960 in the light of continued study and reflection. The revisions have also taken account of teaching experience with the original text at six centers during the past year. We are deeply indebted to the teachers involved for their many helpful comments. We believe that the text can be taught effectively in its present form to average and above-average students who elect 12th grade mathematics.

The text follows generally the outline recommended by the Commission on Mathematics for the first semester of the 12th grade. (For a copy of the Commission's Report, write to College Entrance Examination Board, c/o Educational Testing Service, Box 592, Princeton, New Jersey.) At the same time we have not felt bound to observe the Commission's point of view in all respects.

The question of the time required to teach the various topics included in this course has been the subject of much discussion and has resulted in relegating to the appendices some material not prerequisite to what follows. At the same time we feel that this material is within the grasp of able students and that it will broaden their mathematical backgrounds appreciably. The additional topics provide a means for making differentiated assignments in a class for which such a procedure would be helpful.

A suggested time schedule for this course is as follows:

Chapter 1	2 weeks
Chapter 2	3 weeks
Chapter 3	3 weeks
Chapter 4	5 weeks
Chapter 5	5 weeks
	18 weeks

It is expected that before studying Chapter 5, Circular Functions, students will normally have had as much trigonometry as is contained in the 11th grade S.M.S.G. text. The chapter can be taught, however, without this knowledge. Since it is desirable for motivational reasons that the student should be familiar with the solution of triangles, an appendix has been included which gives a minimum introduction of this sort. The text thereby becomes self-contained. Another appendix includes sufficient material on proving trigonometric identities and solving trigonometric equations to give students practice in these arts if it is deemed necessary.

This commentary includes a general introduction to each chapter of the text and comments on sections where the procedure departs from the customary one. The commentary also includes solutions for all exercises, additional suggested problems, and a set of illustrative test questions at the end of each chapter.

Each chapter of the text contains a miscellaneous set of exercises based on the chapter. Solutions of these exercises have been included in this commentary. Since many of the exercises are time-consuming and will challenge a student's best efforts, teachers should be judicious in assigning them. In general, the miscellaneous exercises are a review of the chapter and an extension of the ideas of the chapter to new and, we hope, interesting situations.

Chapter 1

FUNCTIONS

A Foreword on Sets.

A modern treatment of functions necessarily uses some of the elementary vocabulary and notation of sets, and this text presumes that the students have acquired some familiarity with this language in previous courses. Because this presumption may not always be valid, we include here a summary of the essential information.

Meaning of Set

One of the most natural and familiar ideas of human experience is that of thinking about and identifying a collection of objects by means of a single word. Examples of such words are family, team, flock, herd, deck (of cards), collection, and so forth. We shall use the word set when talking about such a collection, and we shall restrict ourselves to sets that are clearly enough defined so that there is no possible ambiguity about their members. In other words, a set is a collection of objects, described in such a way that there is no doubt as to whether a particular object does or does not belong to the set.

As an illustration, think of the collection of books, pencils, tablets, etc., that is in your desk. You can easily tell whether or not a particular object belongs to this set; if an object is in your desk, then it is a member, or element, of this set; if an object is not in your desk then it is not an element of this set. It is important to understand that it does not matter what objects are in your desk; to be an element of this particular set, the only requirement is that an object be in your desk and not somewhere else.

Describing a Set

We have at our disposal two methods for describing a set: (1) the tabulation method, in which we list or tabulate every element of a set, and (2) the rule method, in which we describe the elements of a set by some verbal or symbolic statement without actually listing its elements. This latter method was used in the preceding paragraph when we defined a set by specifying that it contained all the objects in your desk. Other illustrations of the rule method for defining a set are the following: the set of all boys and girls who attend your school, the set of people who live in your home, the set of books in your school library, or the set of colors your sister is going to use in redecorating her kitchen.

Although the rule method for defining a set will be used predominantly, there are cases in which the only feasible way to define a set is by actually tabulating its elements. This may be because the elements of a set are not required to have anything in common except membership in the set. It is true that most, if not all, of the sets we shall be talking about will consist of things which are naturally assembled together, as, for example, the set of whole numbers. Nonetheless, a set may consist of things which have no obvious relation except that they happen to be grouped together, just as the set of objects which a nine-year old boy calls his "treasure" may consist of a yo-yo, an indian-head penny, a ball made of packed tinfoil, a collection of match books, a dried grasshopper, a pocket knife, and a pack of baseball cards. Perhaps such an example will help to clarify the idea that a set is a collection of things, not necessarily alike in any other respect, and that membership in the set is to be emphasized.

Notation

The notation which is customarily used when defining a set, whether by the tabulation method or the rule method, will be illustrated by another example. Consider this question: what is the set of all coins in your pocket at this moment? (The answer in this case might be the set with no elements -- the empty set!) Suppose that you have three pennies, two nickels, a dime, and a quarter in your pocket, the pennies and nickels being distinguished by different dates. The set called for by the rule is the collection of these seven coins and no others. Using the tabulation method, we symbolize this by writing:

S = {1915 penny, 1937 penny, 1959 penny,
1942 nickel, 1950 nickel, dime, quarter}.

Capital "S" is a name for the set, and the names of the elements of the set are enclosed in the braces. The order in which the elements are listed within the braces does not matter. Alternatively, we may denote this same set by enclosing the rule in braces:

S = {* : * is a coin in your pocket}.

This is read, "S is the set of all * such that * is a coin in your pocket." The colon following the first * is a symbol for the phrase "such that," and the symbol * stands for any unspecified element of the set. We could just as well have used c, or x, or $\overset{\times}{\times}$, so that S = {c : c is a coin in your pocket} is still the set of coins in your pocket. The symbolism {* : * ...} is often called the "set-builder" notation. Some texts use a vertical line instead of the colon, as in {* | * ...}; we prefer the colon for typographical reasons.

In summary, we have illustrated two alternative ways for defining any particular set: (1) the tabulation method, and (2) the rule or set-builder method. As emphasized earlier, each of these methods has the essential characteristic that every object may be classified as either belonging to the set or not belonging to the set. In some cases either method can be used, as we did in describing the set of coins in your pocket. In other situations only one of the two methods may be practical. For example, can you tabulate the elements of this set:

P = {$\overset{\times}{\times}$: $\overset{\times}{\times}$ is a human being who knows that
2 + 2 = 4 but does not know that 5 + 5 = 10}?

Or, can you use the rule method to specify the following rather unusual set without actually listing its elements:

Q = {this book, the moon, your left shoe}?

To indicate membership in a set we use the Greek letter ∈ (epsilon). Thus, if a is a member of the set A, we write a ∈ A. (This may be read, "a is an element of the set A," or "a belongs to the set A," etc.) Likewise, we may wish to indicate that b is not an element of A. In this case we use epsilon with a diagonal line drawn through it, indicating negation, and write $b \notin A$.

Exercises

1. Use both the tabulation method and the rule method to specify the following sets:
    a) the vowels;
    b) the prime numbers less than 20;
    c) the people who live in your house;
    d) the odd multiples of three which are equal to or less than 21;
    e) the two-digit numbers, the sum of whose digits is 8.
2. Represent the following sets by the rule method and tell why the tabulation method may be difficult or impossible:
    a) the set of students in your school;
    b) the integers greater than 7;
    c) the people in your community who found a ten-dollar bill yesterday;
    d) the books in your school library;
    e) the rational numbers between 2 and 3.
3. Find a rule which will define the sets whose elements are tabulated in each of the following:
    a) $A = \{2, 4, 6, 8, 10\}$;
    b) $B = \{-3, -2, -1, 0, 1, 2, 3\}$;
    c) $C = \{1, 4, 9, 16, 25\}$;
    d) $D = \{2, 5, 8, 11, 14, 17\}$;
    e) $E = \{123, 132, 213, 231, 312, 321\}$.

Answers to Exercises on Sets

In all of the following answers, letters, symbols, and names of people or objects as well as their sequence may be different without making the answers incorrect. It is not necessary always to name a set by means of a capital letter.

1. a) V = {a, e, i, o, u} or V = {* : * is a vowel}

   b) P = {2, 3, 5, 7, 11, 13, 17, 19}
   = {p : p is a prime number less than 20}

   For technical reasons, 1 is not considered a prime number. For example, its inclusion would raise a difficulty in the unique factorization theorem.

   c) R = {(insert names of people living in your house)}
   = {a : a is a person who lives in my house}

   d) T = {3, 9, 15, 21}
   = {n : n is an odd multiple of 3 and $n \leq 21$}

   e) N = {17, 26, 35, 44, 53, 62, 71, 80}
   = {x : x is a two-digit integer, the sum of whose digits is 8}

   Note: 08 is not considered a two-digit number in our system.

2. a) S = {s : s is a student in our school}

   b) M = {# : # > 7}

   c) P = {p : p is a person in our community who found a ten-dollar bill yesterday}

   d) B = {b : b is a book in our school library}

   e) F = {f : f is a rational number between 2 and 3}

   Here are three types of sets which are not tabulated.

   (a) and (d) represent extensive and lengthy lists which are available somewhere as completely tabulated sets but usually not duplicated.

   (b) and (e) represent examples of sets which contain an endless number of elements and thus defy listing.

   (c) represents a condition frequently found in mathematics where even though the description is clear and well-defined it still requires a great deal of work or ingenuity to find the elements.

3. a) A = {a : a is a positive even integer less than 12}
   or {a : a is an even natural number less than or equal to 10}

   b) B = {b : b is an integer whose square is less than 10} or
   {b : b is an integer and $-3 \leq b \leq 3$}

c) $C = \{c : c$ is the square of 1, 2, 3, 4, or 5$\}$ or
$\{c : c$ is the square of an integer and $0 < c < 26\}$ or
$\{c^2 : c$ is an integer and $1 \leq c \leq 5\}$

d) $D = \{d : d = 2 + 3n, n$ is an integer, and $0 \leq n \leq 5\}$ or
$\{d : d$ is a number of the form $3n - 1$, and $n = 1, 2, 3, 4, 5$ or $6\}$ or $\{d : d$ is a term in an arithmetic sequence whose first term is 2, whose common difference is 3, and whose last term is $17\}$

e) $E = \{e : e$ is a permutation of the digits 1, 2, and $3\}$
or $\{abc : abc$ is a permutation of $123\}$
or $\{e : e$ is a three-digit integer formed from the digits 1, 2, and 3 without repetition$\}$.

1-1. Functions. Pages 1-7.

A function can be defined in a variety of ways. The definition we have given was selected because it emphasizes the modern point of view of a function as a mapping, a point of view which is a particularly useful one in describing the composition of functions and inverse functions. Another common contemporary practice is to define a function as a set of ordered pairs in which no two distinct pairs have the same first component; this definition is a particularly convenient one in dealing with graphs. These two definitions can be shown to be logically equivalent, and a mathematician would certainly feel free to think of a function in either way. We feel, however, that it may be less confusing to the student if we stick to one definition.

You should be very careful at this stage to insist upon the proper use of functional notation and how to read it. If we write

$$f : x \longrightarrow y$$

then we read,

"The function f that takes (or maps) x into y."

If we write $y = f(x)$, we read

"y is the value of f at x."

e student should not be permitted to say that $y = f(x)$ is a nction. This is a common error of usage. Many mathematicians still use y = f(x) elliptically, but being mathematicians, they understand what they are doing. High school students, however, are apt to be very confused by this and we wish to do everything we can to be clear about the matter. Thus, y = 3 - x is not a "linear function" although it may be used to define one over the real numbers, i.e.

$$f : x \longrightarrow 3 - x.$$

In explaining the function concept you will probably wish to make use of a variety of techniques. The representation as a machine is one. Another approach might be to suggest a function from a domain consisting of the students in the class to a range consisting of the seats in the class and then ask for restrictions on the assignment so that it represents a function. For instance, two different seats could not be assigned to the same student; at least one seat would have to be assigned to each student, etc. Or again, inquire into the possibility of defining a function from the set of students to the set of their weights. Such examples are easy to devise and provide a means of focusing attention on the essential properties of a function. Also it is useful for the student to be aware of the fact that the domain and range of a function need not be numerical. Many times it is useful to consider a function from the real numbers, say, to a set of points in a plane or vice versa (Chapter 5).

It is also helpful to use examples from the sciences. You might ask the students what physicists mean when they say that the length of a metal bar is a function of the temperature of the bar, or that the pressure of a gas at a given temperature is a function of the volume it occupies. Make sure in each case that the students arrive at a function from the real numbers to the real numbers.

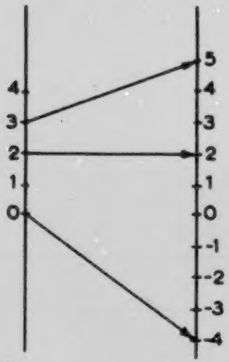
When introducing the idea of a function as a mapping, you should emphasize the point that there cannot be more than one arrow from each element of the domain while there can be any number of arrows to each element of the range. If there is just one arrow to each element in the range, then the function is said to be one-to-one and, as will be seen later, has an inverse.

The concepts of domain and range should be emphasized. It should be made clear that in order to define a function, we must have a domain. (It will prove valuable to the students if from time to time after you have completed the unit you stop and ask for the domain and range of whatever function you may be considering at the time.)

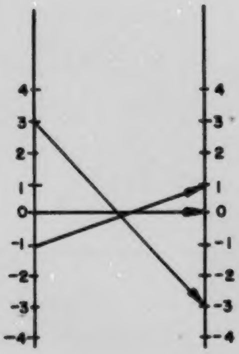
Answers to Exercises 1-1. Pages 7-8.

1. e, because it is multiple valued

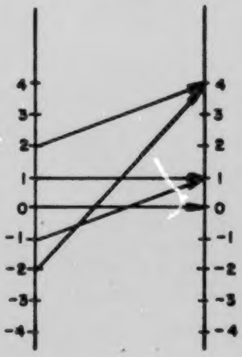
2. a)



c)



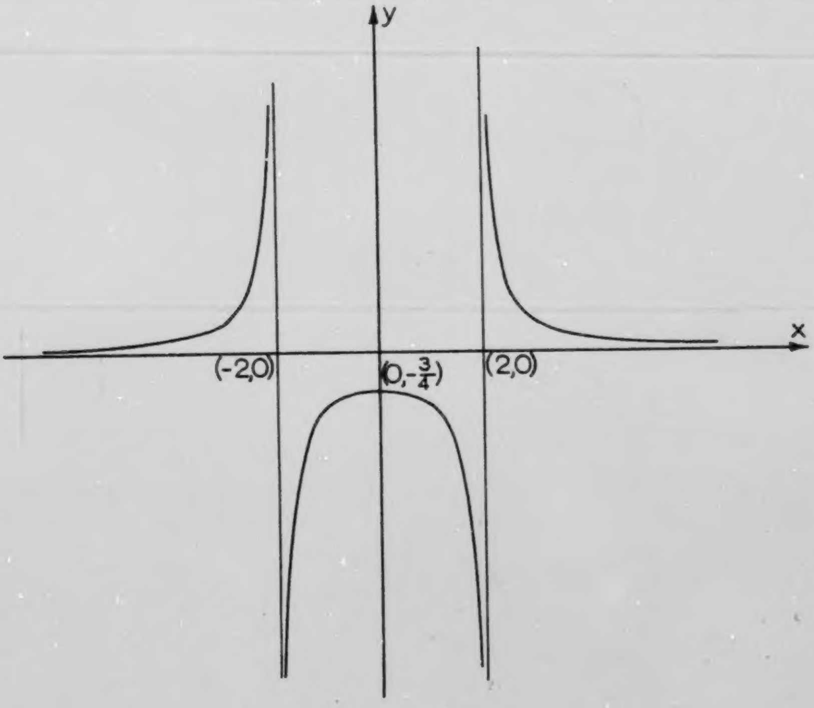
d)



3.	Domain	Range
a.	R	R
b.	R	nonnegative R
c.	nonnegative R	nonnegative R
d.	R except $x = 1$	R except $f(x) = 1$
e.	R except $x = 2$ or -2	R except $-\frac{3}{\pi} < f(x) < 0$.

To find the range in (d), set $y = \frac{x}{x - 1}$ and solve for x: $x = \frac{y}{y - 1}$. This shows that $y \neq 1$.

Do not discuss (e) at length. Simply show graph to look like this:



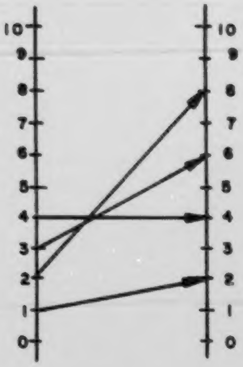
If you test to find values of x such that $-\frac{3}{4} < f(x) < 0$, you will obtain imaginary values for x.

4. a) $f(0) = 1$
   b) $f(-1) = -1$
   c) $f(100) = 201$
   d) $f(\frac{3}{2}) = 4$

5. a) $f(0) = 3$
   b) $f(-1) = 6$
   c) $f(a) = a^2 - 2a + 3$
   d) $f(x - 1) = x^2 - 4x + 6$

6. a) $f(4) = 0$
   b) $f(-5) = 3$
   c) $f(5) = 3$
   d) $f(a) = \sqrt{a^2 - 16}$
   e) $f(a - 1) = \sqrt{a^2 - 2a - 15}$
   f) $f(\pi) = \sqrt{\pi^2 - 16}$

7. D = {1, 2, 3, 4} R = {2, 4, 6, 8}



8. They are not the same function, since g does not include 0 in its domain.

9. a) 4, -4 b) 8 c) 12, -12

1-2. Graph of a Function. Pages 8-12.

The graph is perhaps the clearest means of displaying a function since the story is all there at once. The student can observe the behavior of f for the various portions of the domain, and, in most cases, irregularities are obvious immediately. The difficulty is, of course, that some functions cannot be graphed, as, for example,

$$f : x \rightarrow \begin{cases} 1 & \text{if } x \text{ rational,} \\ 0 & \text{if } x \text{ irrational.} \end{cases}$$

Since high school students are not normally exposed to such functions, however, this is not a very serious obstacle.

The graph might best be introduced by using some function whose behavior is not too obvious. You may, for example, wish to use the "greatest integer contained in" function, which is easily explained and leads to some interesting configurations. We define

$$f : x \longrightarrow [x]$$

as the function which maps x into the greatest integer contained in x. Thus

$$f(1) = 1,\ f(\tfrac{3}{2}) = 1,\ f(\tfrac{1}{2}) = 0,\ f(-\tfrac{3}{2}) = -2, \text{ etc.}$$

The graph of the equation $y = [x]$ is in Figure TC, 1-2a. There are a number of interesting combinations which can be formed with $[x]$.

Figures TC, 1-2b to TC, 1-2d illustrate three of them.

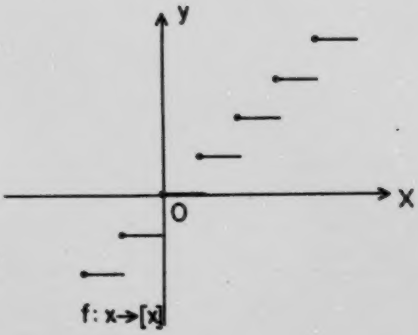


Fig. TC, 1-2a

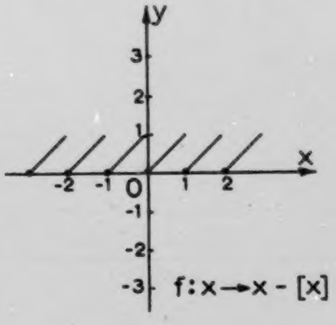


Fig. TC, 1-2b

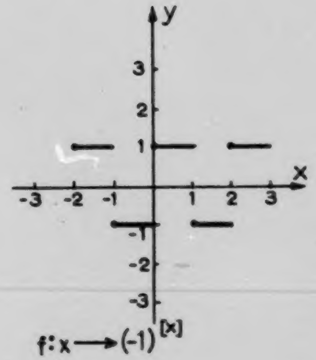


Fig. TC, 1-2c

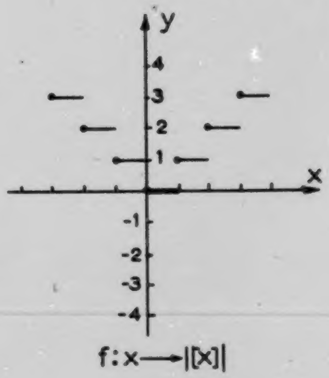


Fig. TC, 1-2d

Furthermore, an infinite checkerboard pattern is given by

$$\{(x, y) : [x] \text{ is even and } [y] \text{ is even}\}.$$

You may also find it helpful to sketch on the blackboard some figures similar to those in Figure 1-2g and to have the students determine whether or not they represent functions by applying the vertical line test. Exercise 2 on page 13 is also a useful type of blackboard exercise and you will probably find it helpful to do one as an illustration before the students attempt to do Exercise 2 themselves.

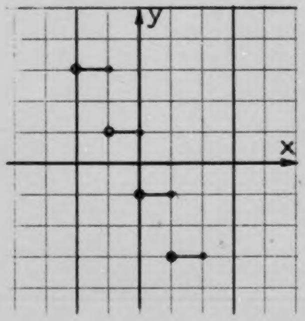
You may want to point out that most of the graphs in the text are incomplete.

Answers to Exercises 1-2. Pages 13-14.

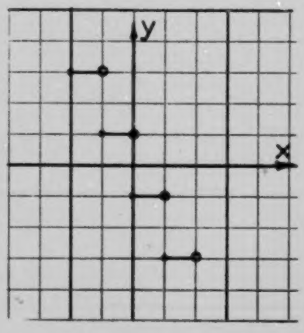
1. a and b

2.

a)

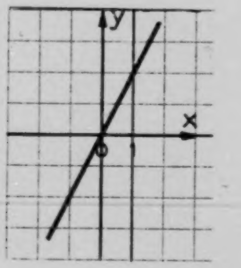


b)

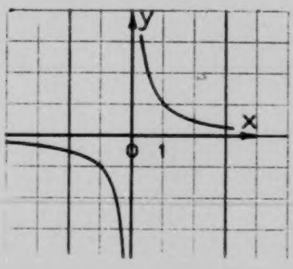


3.

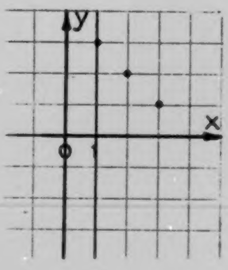
a)



b)

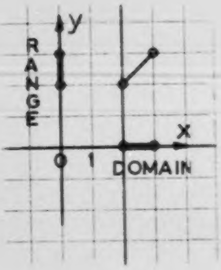


c)

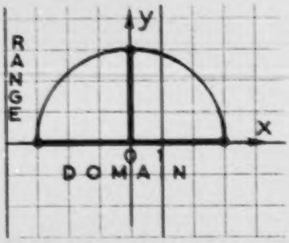


4.

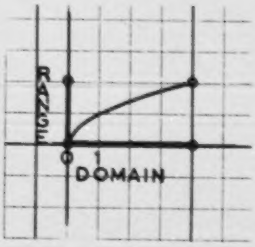
a)



b)



c)



1-3. Constant Functions and Linear Functions. Pages 14-19.

Although the ideas of this section should be familiar to the student, it is unlikely that he will have encountered them in the language of mapping. This section will, therefore, give a review of important material and at the same time valuable practice in the use of functional notation.

Answers to Exercises 1-3. Pages 19-21.

1. (a) Slope = 3 (c) Slope = $-\frac{1}{2}$
   (b) Slope = -2 (d) Slope = $\frac{4}{3}$

2. (a) $f : x \longrightarrow -2x + 6$ (c) $f(x) = -2x + 7$
   (b) $f : x \longrightarrow -2x - 7$ (d) $f(x) = -2x + 13$

3. (a) Slope = $\frac{-3 - 4}{1 - 0} = -7$ (c) Slope = $\frac{-3 - 5}{1 - 5} = 2$
   (b) Slope = $\frac{-3 - 3}{1 - 2} = 6$ (d) Slope = $\frac{-3 + 13}{1 - 6} = -2$

4. (a) $f(x) = 3x - 2$ (c) Not a function.
   (b) $f(x) = -2x - 10$ (d) $f : x \longrightarrow 4$

5. (a) $f: x \longrightarrow -3x + 7$ (c) $f(x) = -3x + 8$
   (b) $f: x \longrightarrow -3x - 3$ (d) $f(x) = -3x - 13$

6. (a) $f(3) = 5$ (c) $f(3) = 4$
   (b) $f(3) = -3$

7. Yes. The slope of the line through P and Q is -2 and the slope of the line through P and S is -2. Two lines through the same point having the same slope coincide.

8. (a) $(100.1 - 100)\left(\frac{39 - 25}{101 - 100}\right) + 25 = .1(14) + 25 = 26.4$
   $f(100.1) = 26.4$
   (b) $.3(14) + 25 = 4.2 + 25 = 29.2$. $f(100.3) = 29.2$
   (c) $f(101.7) = 48.8$
   (d) $f(99.7) = 20.8$

9. (a) $f(53.3) = -44(.3) + 25 = 11.8$
   (b) $f(53.8) = -10.2$
   (c) $f(54.4) = -36.6$
   (d) $f(52.6) = 42.6$

10. $$\begin{cases} 2x + 7y + 1 = 0 \\ x - 2y + 8 = 0 \end{cases}$$

$$\begin{cases} 2x + 7y + 1 = 0 \\ 2x - 4y + 16 = 0 \end{cases}$$

$$11y = 15$$

$$\begin{cases} y = \frac{15}{11} \\ x = -\frac{58}{11} \end{cases}$$

$$x - 3y + 4 = 0$$

$$\text{Slope} = \frac{1}{3}$$

$$y = \frac{1}{3}x + b$$

$$\frac{15}{11} = \frac{1}{3}\left(-\frac{58}{11}\right) + b$$

$$\frac{103}{33} = b$$

$$y = \frac{x}{3} + \frac{103}{33} \quad \text{or}$$

$$11x - 33y + 103 = 0$$

11. The slopes of the lines AB and CD are $\frac{1}{4}$ and the slopes of the lines AD and BC are $-\frac{3}{2}$. Since the opposite sides are parallel (have the same slope), ABCD is a parallelogram.

12. (a) C(4, 8) (b) C(5, -11)

13. $f : x \longrightarrow 2x - 1$
    $f(t + 1) = 2(t + 1) - 1 = 2t + 1$
    $\therefore P(t + 1, 2t + 1)$ is on the graph of f.

14. $f(0) = f(t - 1)$ when $t = 1$. Then $f(0) = 3 \cdot 1 + 1 = 4$
$f(8) = f(t - 1)$ when $t = 9$. Then $f(8) = 3 \cdot 9 + 1 = 28$

15. $f(0) = f(t - 1)$ when $t = 1$. Then $f(0) = 1^2 + 1 = 2$
$f(8) = f(t - 1)$ when $t = 9$. Then $f(8) = 9^2 + 1 = 82$

16. $f(x_1) = mx_1 + b, \quad f(x_2) = mx_2 + b$

$$\begin{aligned} f(x_1) - f(x_2) &= mx_1 + b - (mx_2 + b) \\ &= mx_1 - mx_2 \\ &= m(x_1 - x_2) \end{aligned}$$

Since $m < 0$ and $x_1 < x_2$ or $x_1 - x_2 < 0$,

$$m(x_1 - x_2) > 0$$

$$\therefore f(x_1) - f(x_2) > 0 \text{ or } f(x_1) > f(x_2)$$

1-4. The Absolute-value Function. Pages 21-23.

There are many reasons for studying this function. It is simple but interesting and useful, and most high school students are unfamiliar with it. It is an important tool in many proofs, used not only in the later parts of this book but extensively in more advanced mathematics.

The definition $|x| = \sqrt{x^2}$ is an example of the composition of functions, considered at greater length in the next section; in this case, if $f: x \longrightarrow x^2$ and $g: x \longrightarrow \sqrt{x}$, then the absolute-value function is the compound function gf. You may want to mention this, informally, in anticipation of Section 1-5.

You may have to spend a little time on the definition of the square-root symbol, $\sqrt{}$. The definition is, of course, to a considerable extent arbitrary, but it must be unambiguous if it is to be useful. It would be a great inconvenience if $\sqrt{3}$, for example, represented a number which might be either positive or negative, and, to avoid this inconvenience, we agree that it is positive. We can then represent the negative number whose square is 3, without ambiguity, as $-\sqrt{3}$. Because, for example, $\sqrt{6^2} = 6$, students find it tempting to write $\sqrt{x^2} = x$. This is, of course, false if $x < 0$. The only correct statement that can be made here is, in fact, the second definition of absolute value; Exercise 1 is designed to reinforce this.

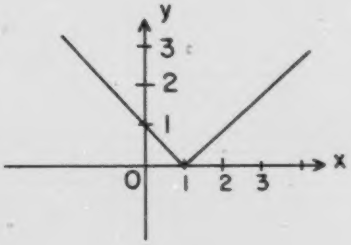
It is convenient, in many applications, to think of $|x|$ as the (undirected) distance on the number line between the origin and the point x. Similarly, $|x - a|$ (or $|a - x|$) is the distance between the point x and the point a. This concept is particularly convenient in problems like Exercises 3 and 4; thus for example, the values of x which satisfy $|x - 5| < 2$ are those values which are less than 2 units from 5, namely all those from 3 to 7. Inequalities such as these appear fairly often in the calculus.

Exercises 1 - 8 are intended to give the pupil an understanding of the meaning of absolute value and some facility in manipulating it. Exercises 9 - 11 are in anticipation of some of the work of Chapter 3.

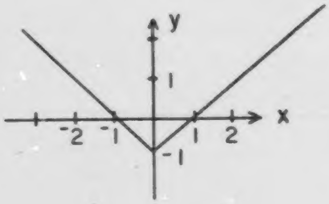
Answers to Exercises 1-4. Pages 23-24.

1. a) $x \geq 0$. b) $x \leq 0$. These follow directly from the definition of the square-root symbol, $\sqrt{\ }$.
2. a) $x - 1 \geq 0$ or $x \geq 1$. b) $x - 1 \leq 0$ or $x \leq 1$.

c)



d)



3. a) The points on the number line 14 units from 0 are 14 and -14. A more formal (and longer) way to arrive at the same result is to note that either $x \geq 0$, in which case $|x| = x$ and the given equation then reads $x = 14$, or

$x < 0$, in which case $|x| = -x$ and the equation reads $-x = 14$ or $x = -14$. Still another approach:

$$|x| = \sqrt{x^2} = 14$$
$$x^2 = 196$$
$$x = \pm 14 \text{ (and both check).}$$

b) $x = 5$ or -9. All three methods discussed under (a) apply here, with $x + 2$ replacing x.

c) Since the absolute value of a number is never negative, it should be clear by inspection that the equation has no roots.

4. a) The problem asks for those points on the number line which are within 1 unit of 2. These are the numbers from 1 to 3, and the solution is therefore $\{x : 1 < x < 3\}$. It is a common practice merely to give the double inequality which defines this set, and state the solution as $1 < x < 3$.

b) Here we must find those points which are more than 2 units from 5; hence $\{x : x < 3 \text{ or } x > 7\}$.

c) $\{x : -4.2 < x < -3.8\}$, as in (a). Note that $|x + 4| = |x - (-4)|$ is the distance between x and -4.

d) By Theorem 1-1, $|2x - 3| = 2|x - 1.5|$; hence the given inequality becomes

$$2|x - 1.5| < 0.04$$
$$|x - 1.5| < 0.02.$$

and the solution, as in (a), is $\{x : 1.48 < x < 1.52\}$.

e) $\{x : -1.28 < x < -1.22\}$, as in (d).

5. If $x \geq 0$, then $|x| = x$ and $x \cdot |x| = x^2$; if $x < 0$, then $x \cdot |x| = x(-x) = -x^2 < 0 < x^2$.

6. In Theorem 1-2, replace b by $-b$; since $|-b| = |b|$, the desired result follows immediately.

7. If $a > b$, then $|a - b| = a - b$ and the given expression becomes

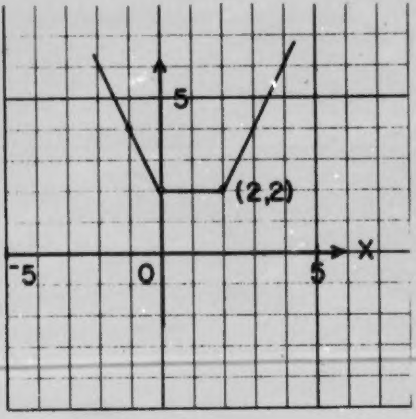
$$\tfrac{1}{2}(a + b + a - b) = a.$$

If $a < b$, then $|a - b| = -a + b$ and the given expression becomes

$$\tfrac{1}{2}(a + b - a + b) = b.$$

The lesser of a and b is given by $\frac{1}{2}(a + b - |a - b|)$. Trial is as good a way as any to get this.

8. If $x < 0$, then $|x| = -x$, $|x - 2| = -x + 2$, and $y = -2x + 2$.
If $0 \leq x < 2$, then $|x| = x$, $|x - 2| = -x + 2$, and $y = 2$.
If $x \geq 2$, then $|x| = x$, $|x - 2| = x - 2$, and $y = 2x - 2$.
Hence we get



9. $|x^2 + 2x| \leq |x^2| + |2x|$ by Theorem 1-2

$\leq |x^2| + 2|x|$ by Theorem 1-1

$< |x| + 2|x| = 3|x|$ by the inequality given in the Exercise.

10. Multiplying both sides of $x < k$ by the positive number x gives $x^2 < kx$. Then, proceeding as in Exercise 9, we get $|x^2 - 3x| \leq |x^2| + |-3x| = |x^2| + 3|x| < 0.1|x| + 3|x| = 3.1|x|$ if $|x| < 0.1$.

11. $|x| < 0.001$, or $-0.001 < x < 0.001$. This result can be established as in Exercise 10.

Answers to Exercises 1-5. Pages 28-29.

1. a) -1 b) $+1$ c) 5
d) 63 e) $x^2 + 4x + 3$ f) $x^2 + 1$
g) $x + 5$

2. a) $acx + ad + b$ b) $acx + cb + d$

d) Theorem: The slope of either composite of two linear functions is equal to the product of the slopes of the two linear functions.

3. a) $1, \quad -3, \quad 8$

b) If $f : x \longrightarrow \frac{1}{x}$, then $ff : x \longrightarrow x$ for all $x \neq 0$.

4. a) $fj : x \longrightarrow x + 2$ $jf : x \longrightarrow x + 2$

b) $g : x \longrightarrow x - 2$

c) $h : x \longrightarrow x - 2$

5. a) $(fg)(x) = x^6$ and $(gf)(x) = x^6$

b) $(fg)(x) = (gf)(x) = x^{mn}$

6. a) $(f \cdot g)(x) = x^5$

b) $(f \cdot g)(x) = x^{m + n}$

7. a) $(f \cdot g)(x) = x^2 - x - 6$

b) $((f \cdot g)h)\ (x) = x^4 - x^2 - 6$

c) $(fh)(x) = x^2 + 2$

d) $(gh)(x) = x^2 - 3$

e) $((fh) \cdot (gh))\ (x) = x^4 - x^2 - 6$

8. The result is true and can be proved as follows: Given 3 functions, $f : x \longrightarrow f(x)$, $g : x \longrightarrow g(x)$, and $h : x \longrightarrow h(x)$, we wish to show that

$$(f \cdot g)h = (fh) \cdot (gh).$$

(It is assumed throughout that f, g, and h are being discussed for all x in the intersection of their domains.)

By definition: $(f \cdot g)(x) = f(x) \cdot g(x)$,

hence

$$\begin{aligned} ((f \cdot g)h)(x) &= (f \cdot g)(h(x)) = f(h(x)) \cdot g(h(x)) \\ &= \big((fh) \cdot (gh)\big)(x) \end{aligned}$$

9. The theorem is false, as the following counterexample shows: Take for both g and h the identity function $x \longrightarrow x$ and for f the function $x \longrightarrow x + 1$. Then $(g \cdot h)(x) = x^2$ and $(f(g \cdot h))\ (x) = x^2 + 1$, but $(fg)(x) = (fh)(x) = x + 1$, and therefore $((fg) \cdot (fh))\ (x) = (x + 1)^2 \neq x^2 + 1$.

10. $(f + g)h = fh + gh$, since $(f + g)(x) = f(x) + g(x)$, and if x is replaced by $h(x)$, we obtain $(f + g)(h(x)) = f(h(x)) + g(h(x))$. But $f(g + h) \neq fg + fh$; this can be shown using as counterexample the functions suggested under Exercise 9.

11. Take $f(x) = m_1x + b_1$, $g(x) = m_2x + b_2$, $h(x) = m_3x + b_3$.

Then $(gh)(x) = g(h(x)) = m_2(m_3x + b_3) + b_2 = m_2m_3x + m_2b_3 + b_2$,

and $(f(gh))(x) = f((gh)(x)) = m_1(m_2m_3x + m_2b_3 + b_2) + b_1$

$$= m_1m_2m_3x + m_1m_2b_3 + m_1b_2 + b_1.$$

Similarly, $(fg)(x) = f(g(x)) = m_1(m_2x + b_2) + b_1$

$$= m_1m_2x + m_1b_2 + b_1$$

and $((fg)h)(x) = (fg)(h(x)) = m_1m_2(m_3x + b_3) + m_1b_2 + b_1$

$$= m_1m_2m_3x + m_1m_2b_3 + m_1b_2 + b_1$$

$$= (f(gh))(x).$$

Since this result is valid for all $x \in R$, it follows that

$$(fg)h = f(gh).$$

1-6. Inversion. Pages 30-33.

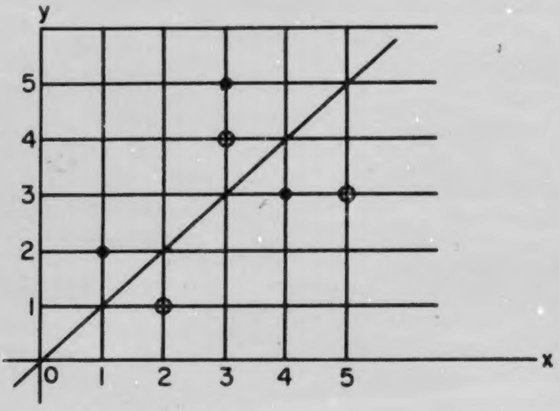
It is frequently helpful in conveying the idea of an inverse to consider functions with a finite domain. Thus, let $f : x \longrightarrow y$ be described by the table

x	1	3	4
y	2	5	3

Domain of $f = \{1, 3, 4\}$

Range of $f = \{2, 5, 3\}$

which we may represent on a graph as dots.



f^{-1} undoes what f does. Thus, if f sends 1 into 2, f^{-1} sends 2 right back to 1. Hence, the domain of f^{-1} is the range of f, $\{2, 5, 3\}$, and the range of f^{-1} is the domain of f. We may write the table for f^{-1} as

x	2	5	3
y	1	3	4

pictured on the above graph as circles. Note the symmetry of f and f^{-1} with respect to the graph of $y = x$.

When the expression on the right side of the arrow is simple, f^{-1} may be easily obtained from f. Thus, if $f\colon x \longrightarrow 3x - 2$, then $f^{-1}\colon x \longrightarrow \frac{x+2}{3}$. f corresponds to the instructions "multiply by 3 and then subtract 2." To "undo" this and obtain f^{-1} we add 2 and then divide by 3. This is all well and good for simple functions. However, this approach no longer works if $f\colon x \longrightarrow \frac{x+1}{x+2}$.

If we write $f : x \longrightarrow y$ where $y = \frac{x+1}{x+2}$, when we seek f^{-1} we want to find the value of x that is associated with a particular y. Hence, we solve $y = \frac{x+1}{x+2}$ for x, obtaining $x = \frac{2y-1}{-y+1}$. We then usually write $f^{-1}\colon x \longrightarrow \frac{2x-1}{-x+1}$ using "x" in place of "y".

We could check this result quickly by taking a specific value for x, say $x = 0$, and seeing whether f^{-1} undoes what f does. Thus, $f\colon 0 \longrightarrow \frac{0+1}{0+2} = \frac{1}{2}$ or $f(0) = \frac{1}{2}$. f^{-1} should send $\frac{1}{2}$ back to 0. Let us see if it does.

$f^{-1}\colon \frac{1}{2} \longrightarrow \dfrac{2(\frac{1}{2}) - 1}{-(\frac{1}{2}) + 1} = \dfrac{1 - 1}{\frac{1}{2}} = 0$ and it does. In general

we have $(f^{-1}f)(x) = f^{-1}(f(x)) = f^{-1}(\frac{x+1}{x+2})$

$$= \frac{2\frac{x+1}{x+2} - 1}{-\frac{x+1}{x+2} + 1} = \frac{2(x+1) - (x+2)}{-(x+1) + (x+2)}$$

$$= \frac{2x + 2 - x - 2}{-x - 1 + x + 2} = \frac{x}{1}$$

$$= x.$$

Moreover, $(ff^{-1})(x) = f(f^{-1}(x)) = f(\frac{2x - 1}{-x + 1})$

$$\text{or } (ff^{-1})(x) = \frac{\frac{2x - 1}{-x + 1} + 1}{\frac{2x - 1}{-x + 1} + 2} = \frac{2x - 1 + (-x + 1)}{2x - 1 + 2(-x + 1)}$$

$$= \frac{x}{1} = x.$$

In general, if $f: x \longrightarrow y = f(x)$, then solve the equation $y = f(x)$ for x in terms of y. This enables us to associate with a given y its x-partner and thus reveals the inverse, f^{-1}. Of course, if f does not have an inverse, the expression obtained for x in terms of y will reveal this.

Answers to Exercises 1-6. Page 33.

1. a) $x \longrightarrow x + 7$

b) $x \longrightarrow \frac{x - 9}{5}$

c) $x \longrightarrow \frac{1}{x}$

2. a) $x = y + 7$ compared with $x \longrightarrow x + 7$

b) $x = \frac{y - 9}{5}$ compared with $x \longrightarrow \frac{x - 9}{5}$

c) $x = \frac{1}{y}$ compared with $x \longrightarrow \frac{1}{x}$

3. Let the number be x; then the various instructions given can be represented by the functions f_1 to f_7, as follows:

$$f_1 : x \longrightarrow 5x$$
$$f_2 : x \longrightarrow x + 6$$
$$f_3 : x \longrightarrow 4x$$
$$f_4 : x \longrightarrow x + 9$$
$$f_5 : x \longrightarrow 5x$$
$$f_6 : x \longrightarrow x - 165$$
$$f_7 : x \longrightarrow \frac{x}{100}$$

Then

$$f_7f_6f_5f_4f_3f_2f_1 : x \longrightarrow \frac{5(4(5x + 6) + 9) - 165}{100} = x.$$

Answers to Miscellaneous Exercises of Chapter 1. Pages 35-37.

1. a)
2. a) Function d) Not a function
   b) Function e) Not a function
   c) Function with inverse f) Function.
3. $f : x \longrightarrow 2$ or $y = 2$.
4. When $a = b = 0$ and $c \in R$.
5. The point of intersection is $(\frac{1}{5}, -\frac{7}{5})$, so $f : x \longrightarrow -\frac{7}{5}$.
6. The point of intersection is $(\frac{b-4}{a-5}, \frac{ab-20}{a-5})$, provided $a \neq 5$. If $a = 5$, they will be parallel, or if also $b = 4$, they will coincide.
7. $f_1: x \longrightarrow x + 3$ and $f_2: x \longrightarrow -x - 3$.
8. $10x + y - 7 = 0$ or $y = 7 - 10x$ and $m = -10$. This means for each unit increase in x, y decreases by 10. If x increases from 500 to 505, y decreases 50. If y decreases from -500 to -505, x increases .5 or $\frac{1}{2}$.
9. Slope of line $= \frac{3-1}{2-(-1)} = \frac{2}{3}$ and line is $y = \frac{2}{3}x$.
10. Point of intersection is $(0, k)$, so line is $y = \frac{5}{6}x + k$.
11. $6x + 3y - 7 = 0 \Longrightarrow y = \frac{7}{3} - 2x$

    $y = -2x + 3 \Longrightarrow y = 3 - 2x$

    Lines are parallel and $\frac{2}{3}$ of a unit apart on the y-axis. Half of this distance is $\frac{1}{3}$ so the line is $y = \frac{8}{3} - 2x$.
12. $2y = x + 3 \Longrightarrow y = \frac{x}{2} + \frac{3}{2}$. Slope $= \frac{1}{2}$.

    Slope of $\perp$ is -2. So line is $y = -2x + 18$.
13. $y = f_1(t) = t - 10$ when t is in minutes.
    $y = f_2(t) = 60(t - \frac{1}{6})$ when t is in hours. Domain of t for f_1 is $\{t : t \geq 10$ and t is a natural number.$\}$; for f_2 it is $\{t : t \geq \frac{n}{60}$ and $n \geq 10$ and n is a natural number.$\}$
14. $\left.\begin{array}{l}\text{(a) AC: } y = \frac{7}{12}x \\ \text{(b) BD: } y = 14 - \frac{7}{4}x\end{array}\right\}$ (c) Intersection is point $(6, 3\frac{1}{2})$

15. (a) AC: $y = \frac{y_2}{x_2} x$

(b) BD: $y = \frac{y_2}{x_2 - 2x_1}(x - x_1)$

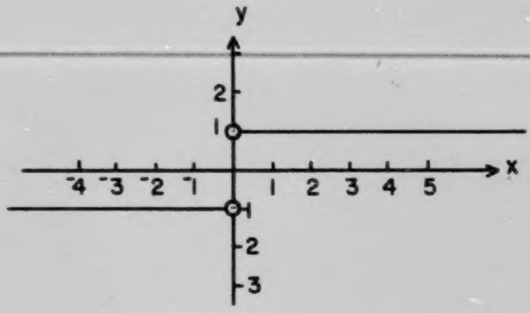
(c) Intersection is point $\left(\frac{x_2}{2}, \frac{y_2}{2}\right)$

16. $(f(gh))(x) = f(g(h(x))) = f(g(c)) = f(b) = a,$
$(f(hg))(x) = f(h(g(x))) = f(h(b)) = f(c) = a.$
But gh: $x \longrightarrow b$ and hg: $x \longrightarrow c$, so these two are different unless $b = c$.

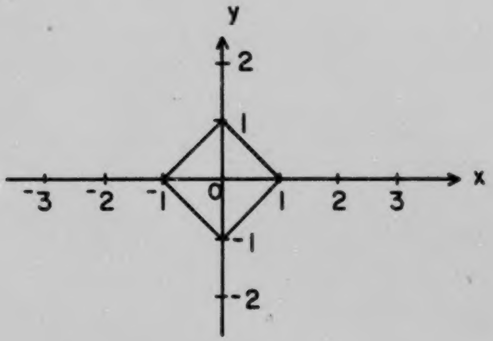
17. $f^{-1}: x \longrightarrow \frac{x - b}{m}$.

18. Any constant function $x \longrightarrow c$, or the identity function $x \longrightarrow x$, or the absolute-value function $x \longrightarrow |x|$.

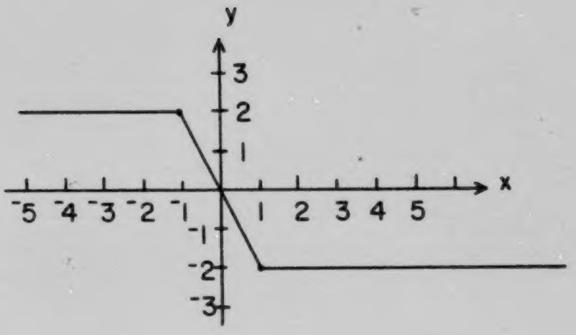
19. a)



b) Note that neither absolute value can exceed 1; hence x is limited to $\{x : -1 \le x \le 1\}$ and y is similarly restricted.



c)



20. $(fg)(x) = 2(3x + k) - 5 = 6x + 2k - 5$

$(gf)(x) = 3(2x - 5) + k = 6x - 15 + k$

$$6x + 2k - 5 = 6x - 15 + k$$

$$k = -10$$

21.

	Domain	Range
f	R	$\{y : y \geq 0\}$
g	$\{x : \lvert x\rvert \leq 4\}$	$\{y : 0 \leq y \leq 4\}$

The intersection of the range of f and the domain of g is $\{y : 0 \leq y \leq 4\}$. The elements of this set are the images, under the mapping f, of $\{x : |x| \leq 2\}$, which is therefore the domain of gf.

The intersection of the range of g and the domain of f is the range of g itself; hence the domain of fg is the domain of g, that is, $\{x : |x| \leq 4\}$.

Illustrative Test Questions for Chapter 1

The following questions (with answers appended) have been included in order to assist you in the preparation of tests and quizzes. The order of the items is approximately the same as the order in which the various concepts being tested appear in the text. This means that you can use selected problems from this list before the chapter has been completed.

For a short quiz, one or two problems from this list would be sufficient; a full period (40 to 50 minutes) test might contain anywhere from five to ten of the problems. It would be a mistake to give all the questions as a chapter test unless at least two class periods were planned for it.

1. Given $f: x \longrightarrow x^2 + 2$, find

   a) $f(3)$ b) $f(6)$ c) $f(\frac{3}{2})$

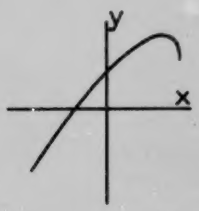
2. Find the domain of f if it is the largest set of real numbers that f maps into real numbers, and find also the corresponding range:

   a) $f: x \longrightarrow \sqrt{x - 1}$.

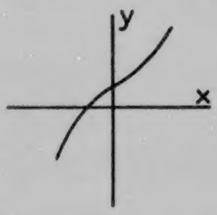
   b) $f: x \longrightarrow \frac{x + 2}{x + 1}$

3. Which of these could be the graph of
   a) a function $f: x \longrightarrow y$,
   b) a function $f: y \longrightarrow x$?

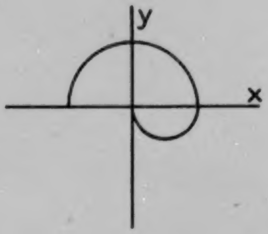
(1)



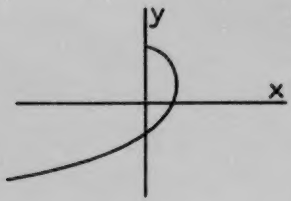
(2)



(3)

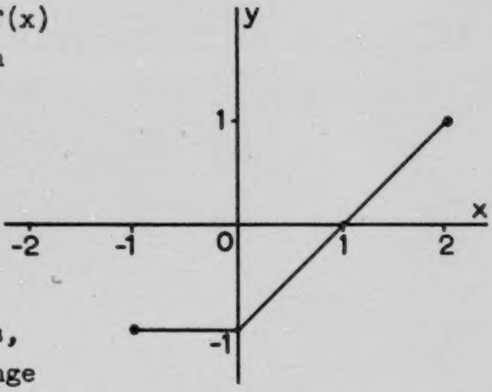


(4)



4. Given the function $f: x \longrightarrow f(x)$ sketched at the right, sketch the graphs of the following functions:

   a) $g: x \longrightarrow -f(x)$
   b) $h: x \longrightarrow -f(-x)$
   c) $k: x \longrightarrow f(|x|)$
   d) $m: x \longrightarrow |f(x)|$



5. Graph the following functions, indicating the domain and range on the appropriate axes:

   a) $f: x \longrightarrow 1 + \sqrt{x},\ x > 1$
   b) $g: x \longrightarrow 1 + |x - 1|,\ -1 \leq x \leq 2.$

6. Graph: $|x| + 2|y| = 4.$

7. Solve: a) $|x + 3| < 0.2$
   b) $|2x - 5| < 0.1.$

8. Find a linear function f such that $f(2) = 3$ and $f(3) = 2f(4)$.

9. If a linear function f has slope $\frac{3}{2}$ and if $f(2) = -3$, find $f(7)$.

10. What is the slope of a linear function f if $f(5) - f(2) = 4$?

11. Find the linear function whose graph passes through all points with coordinates of the form $\big((t + 3)(t - 2),\ (t + 4)(t - 3)\big)$.

12. Find the value of k for which $(k, 2k)$ lies on the line through $(3, -2)$ and $(5, 4)$.

13. Given $f: x \longrightarrow 3x + 1$ and $g: x \longrightarrow x^2 - 2$, find the function $fg - gf$.

14. Given $f: x \longrightarrow 2x + 1$ and $g: x \longrightarrow x^2 - 1$, solve the equation $(gf)(x) = 0$.

15. Given $f: x \longrightarrow 3x + 5$ and $g: x \longrightarrow 2x + k$, find k if $(gf)(x) = (fg)(x)$ for all $x \in R$.

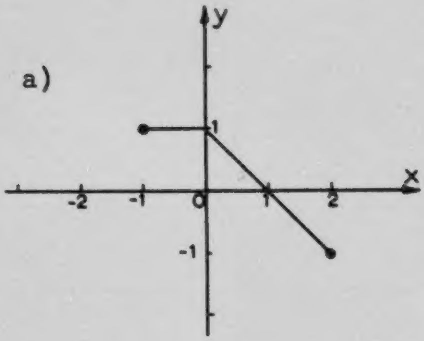
Answers to Illustrative Test Questions.

1. a) 11 b) 38 c) $\frac{17}{4}$
2. a) Domain: $\{x : x \geq 1\}$. Range: $\{y : y \geq 0\}$.
   b) Domain: $\{x : x \neq -1\}$. Range: $\{y : y \neq 1\}$.

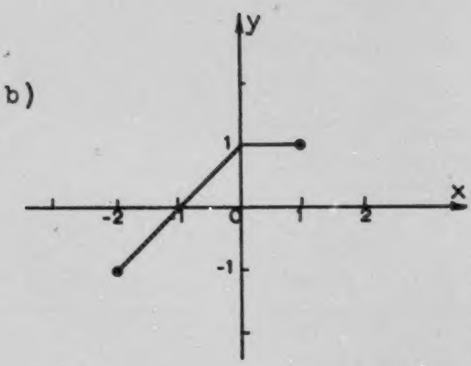
3. a) (1), (2).
   b) (2), (4).

4.

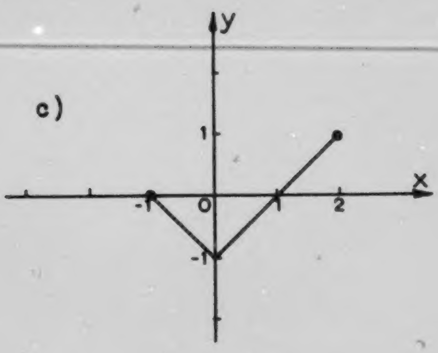
a)



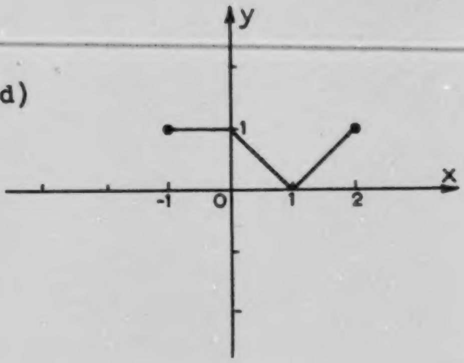
b)



c)

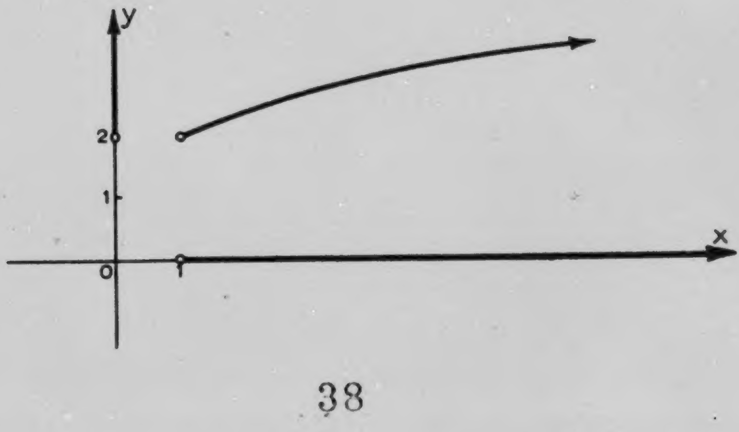


d)

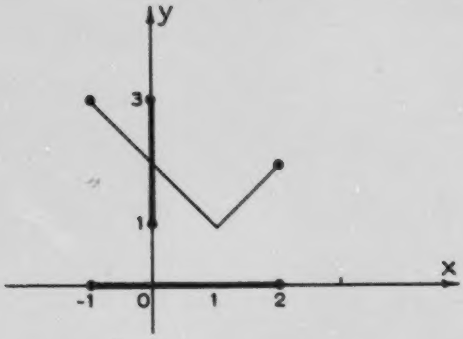


5.

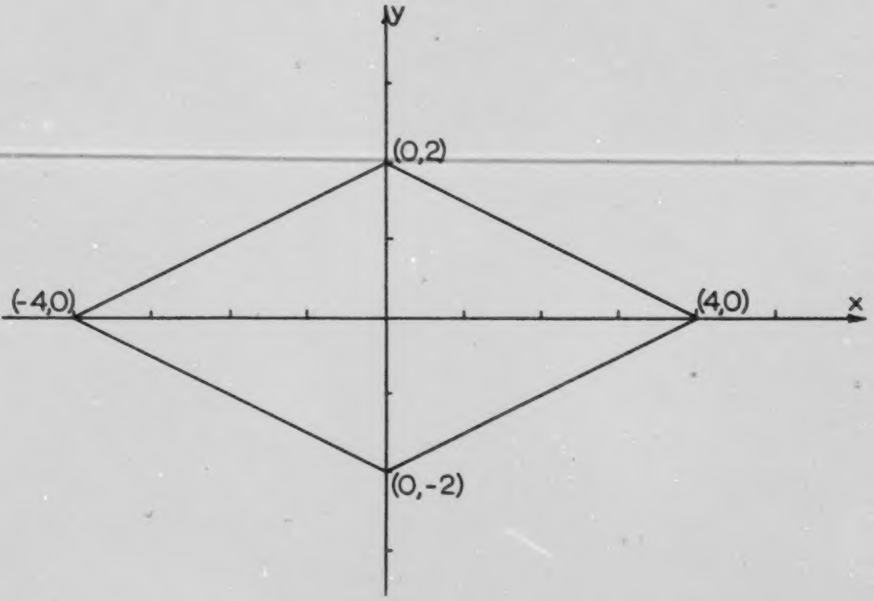
a)



b)



6.



7. a) $-3.2 < x < -2.8$

b) $2.45 < x < 2.55$

8. $x \longrightarrow -x + 5$

9. $f(7) = \frac{9}{2}$

10. $m = \frac{4}{3}$

11. $x \longrightarrow x - 6$

12. $k = 11$

13. fg-gf: $x \longrightarrow -6x^2 - 6x - 4$

14. $x = 0, -1$

15. $k = \frac{5}{2}$

Chapter 2

POLYNOMIAL FUNCTIONS

Introduction.

This is the first of two chapters on polynomial functions. It covers material on the solution of polynomial equations usually included in a course in advanced algebra, although it differs from conventional treatments in certain respects. (1) It uses somewhat more precise terminology than is customary. For example, we distinguish between "polynomial" and "polynomial function". (2) The treatment of synthetic division, or as we prefer to call it, synthetic substitution, is an application of the distributive property of the real numbers and leads directly to the Remainder Theorem without the use of long division. (3) For the approximation of irrational roots we use the Location Theorem and linear interpolation rather than Horner's method. In Chapter 3 we introduce Newton's method as a more powerful means of approximating irrational roots.

Some historical information has been included in the text with detailed references to generally accessible books. This subject offers an unusual opportunity to give the student an interesting and understandable introduction to portions of the history of mathematics.

The Appendix contains a section on the importance of polynomials and includes a treatment of the Lagrange interpolation formula. This work is suitable for a longer course or for superior students.

2-1. Introduction and Notation. Pages 39-43.

This section contains definitions of "zero of a function" and "polynomial function" in terms of the concepts developed in Chapter 1, and it also includes an overview of the history of the problem of determining the zeros of a given polynomial function.

It is important to emphasize again the distinction made in Chapter 1 between a function and the means of defining a function.

Thus, a polynomial expression

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$$

defines a polynomial function

$$f:\ x \longrightarrow f(x).$$

From this point of view it is incorrect to speak of $f(x)$ as a function. On the other hand, we do occasionally use the word "polynomial" for the sake of verbal simplicity when it is clear from the context that we mean "polynomial function". The distinction is likely to be more troublesome for teachers accustomed to conventional treatments of algebra than to their students.

2-2. Evaluation of $f(x)$ at $x = c$. Pages 43-47.

So-called "synthetic division" is presented as an application of the distributive principle that is efficient and easily justified. The process here developed is used in computing with machines since the only operations required are successive multiplications and additions rather than raising to powers.

Answers to Exercises 2-2. Pages 47-48.

1.

1	0	0	1	-3	
1	-2	4	-7	11	-2
1	1	1	2	-1	1
1	3	9	28	81	3

2.

-3	1	1	-2	
-3	4	-3	1	-1
-3	10	-29	85	-3
-3	1	1	-2	0
-3	-5	-9	-20	2
-3	-11	-43	-174	4

3.

3	-2	0	1	
3	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{7}{8}$	$\frac{1}{2}$
3	-1	$-\frac{1}{3}$	$\frac{8}{9}$	$\frac{1}{3}$
3	4	8	17	2

4.

6	-5	-17	6	
6	4	-11	$-\frac{21}{2}$	$\frac{3}{2}$
6	-2	-18	-3	$\frac{1}{2}$
6	-8	-13	$\frac{25}{2}$	$-\frac{1}{2}$
6	-3	-18	0	$\frac{1}{3}$
6	7	-3	0	2

5.

6	-29	37	-12	
6	-29	37	-12	0
6	-23	14	2	1
6	-17	3	-6	2
6	-11	4	0	3
6	-5	17	56	4

6. $k = 3$. Use direct or synthetic substitution.

7. 5. This exercise illustrates a case in which synthetic substitution is easier.

8. 1002. This exercise illustrates a case in which direct substitution is easier.

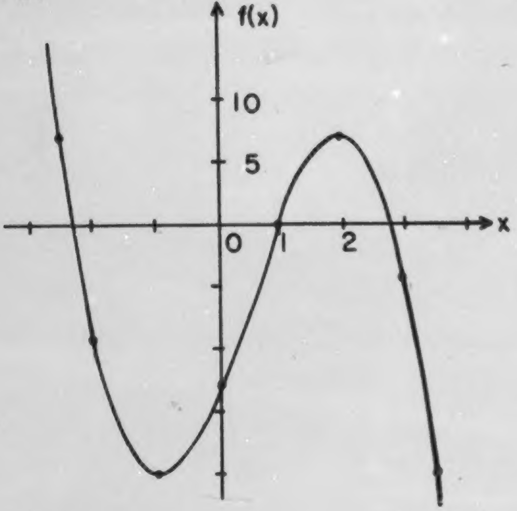
2-3. Graphs of Polynomial Functions. Pages 48-53.

The purpose of this section is to give students a first introduction to the graphs of polynomial functions of degree greater than 2, and, incidentally, to reinforce the technique of synthetic substitution. Plotting a large number of points is not a very efficient way to obtain the graph of a polynomial function, but we believe that it is a helpful first step leading ultimately to proficiency in sketching graphs by means of intercepts, maximum and minimum points, and points of inflection, to be developed in Chapter 3.

The continuity of polynomial functions is assumed, but we feel that teachers should recognize the importance of this concept and be able to satisfy students on an intuitive level that the graph of a polynomial function contains no holes or breaks. The simple but tedious expedient of evaluating $f(x)$ for any suggested real number $x = c$, and also for values of x near c, should convince students of the reasonableness of the assumption.

Answers to Exercises 2-3. Pages 53-54.

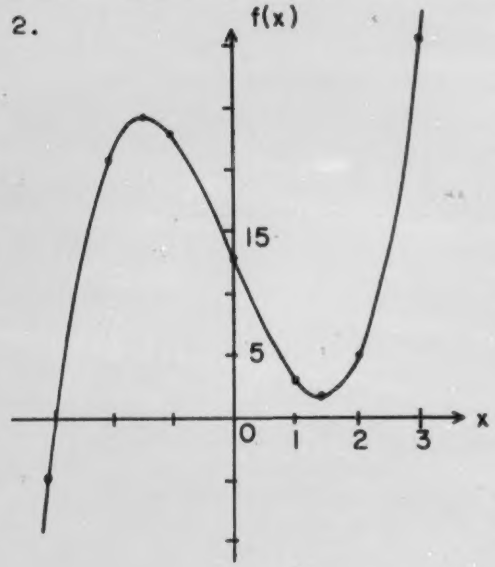
1.



f: $x \longrightarrow -2x^3 + 3x^2 + 12x - 13$

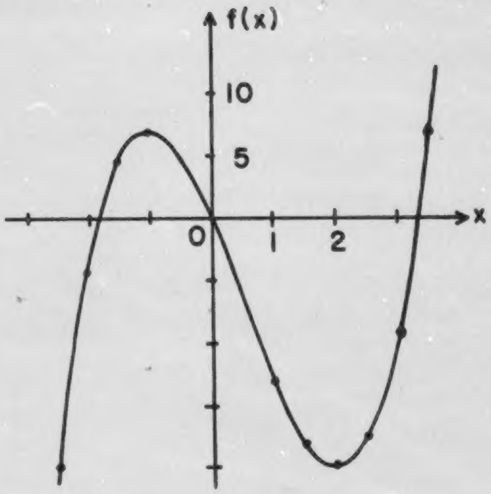
This is the graph of Figure 2-3b inverted. It intersects the x-axis at the same points.

2.



f: $x \longrightarrow 2x^3 - 12x + 13$

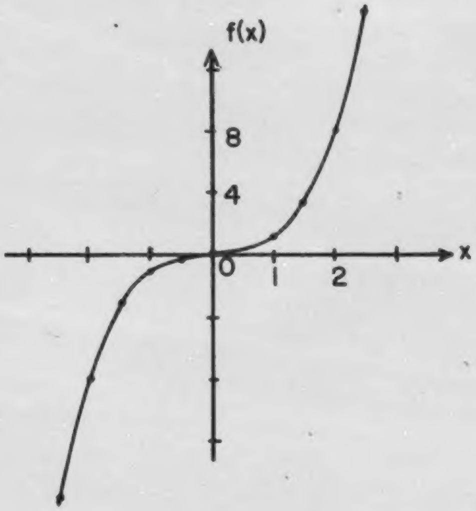
3.



f: $x \longrightarrow 2x^3 - 3x^2 - 12x$

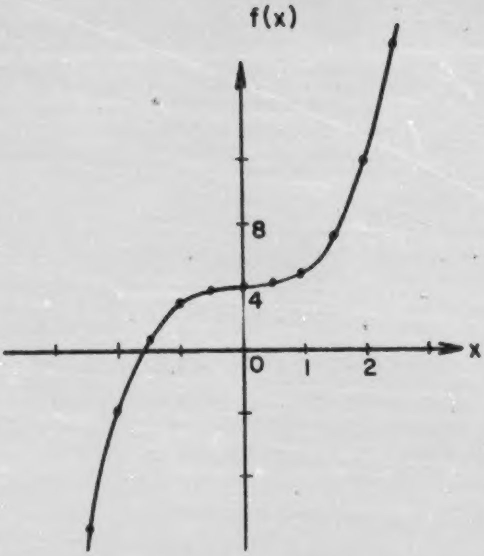
This is the graph of Figure 2-3b moved down 13 units.

4.



f: $x \longrightarrow x^3$

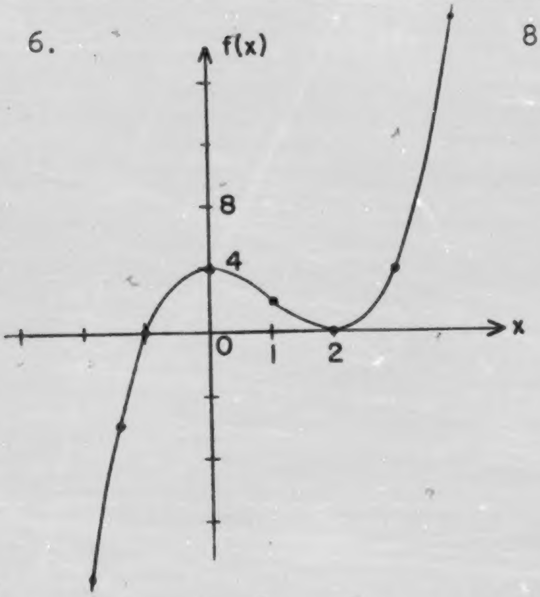
5.



f: $x \longrightarrow x^3 + 4$

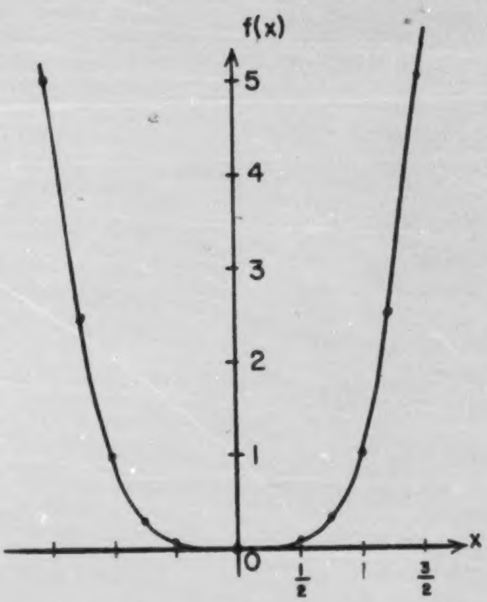
This is the graph of Exercise 4 moved up 4 units.

6.



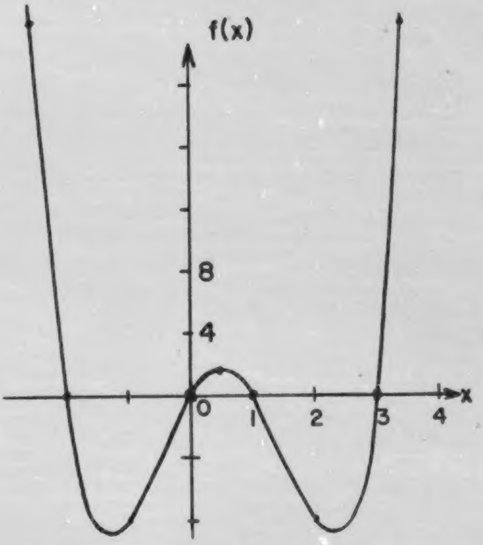
f: $x \longrightarrow x^3 - 3x^2 + 4$

7.



f: $x \longrightarrow x^4$

8.



f: $x \longrightarrow x^4 - 2x^3 - 5x^2 + 6x$

2-4. Remainder and Factor Theorems. Pages 54-57.

In this section the Remainder and Factor Theorems are developed as an outgrowth of the process of synthetic substitution. The Factor Theorem is applied as a testing device.

In conventional treatments of the Remainder Theorem, the conclusion is a direct consequence of the division algorithm; namely, of the fact that

$$\text{Dividend} = (\text{Divisor}) \cdot (\text{Quotient}) + \text{Remainder}.$$

In these same treatments, the technique which we call "synthetic substitution" is introduced as an abbreviated method for the long division of a polynomial by $x - c$. Thus, the relationships among long division, synthetic division, and the Remainder Theorem are established.

Long division of polynomials, however, may be an unfamiliar process. It is, nevertheless, important to be able to obtain the quotient as well as the remainder for such a division. Synthetic substitution, as an application of the distributive property, enables us to do this without recourse to long division. If students need additional evidence of this fact, you should encourage them to check the results of synthetic substitution of c by actually multiplying $x - c$ by the quotient and then adding the remainder $f(c)$ to obtain the original polynomial. We illustrate by an example.

Example. Find the quotient and remainder when

$$2x^3 - 6x^2 - x + 7 \text{ is divided by } x - 2,$$

and check your results by multiplication.

Solution. To divide the given polynomial by $x - 2$, we use synthetic substitution of 2:

$$\begin{array}{rrrrl} 2 & -6 & -1 & 7 & \underline{|2} \\ & 4 & -4 & -10 & \\ \hline 2 & -2 & -5 & \multicolumn{1}{|r}{-3} & \end{array}$$

The quotient is $2x^2 - 2x - 5$ and the remainder is -3. Therefore,

$$2x^3 - 6x^2 - x + 7 = (x - 2)(2x^2 - 2x - 5) - 3.$$

To check, we perform the indicated multiplication:

$$(x - 2)(2x^2 - 2x - 5) = 2x^3 - 6x^2 - x + 10.$$

Adding the remainder -3 gives the original polynomial.

Answers to Exercises 2-4. Pages 57-58.

1.

	$q(x)$	$f(c)$
a)	$3x^2 + 10x + 10$	5
b)	$x^2 + 2$	6
c)	$-2x^3 - 3x^2 - 9x - 21$	-73
d)	$2x^2 - 2x + 4$	0

2.

	Quotient	Remainder
a)	$x^2 + 6x + 5$	7
b)	$x^2 + x - 2$	0
c)	$x^2 + 2x - 1$	-1

3. $q(x)$ is of degree $n - m$.
   $r(x)$ is of degree less than m.

4. Exercise 4 in Section 2-2 has $x - 2$ as a factor and $x - \frac{1}{3}$ or $3x - 1$ as a factor.
   Exercise 5 has $x - 3$ as a factor.

5.

1	4	1	-6	
1	7	22	60	3
1	6	13	20	2
1	5	6	0	1
1	4	1	-6	0
1	3	-2	-4	-1
1	2	-3	0	-2
1	1	-2	0	-3

$$\begin{aligned} f(x) &= x^3 + 4x^2 + x - 6 \\ &= (x - 1)(x + 2)(x + 3) \end{aligned}$$

6.

2	1	-5	2	
2	-3	1	0	-2
2	-1	-4	6	-1
2	1	-5	2	0
2	3	-2	0	1
2	5	5	12	2
2	2	-4	0	$\frac{1}{2}$

$$\begin{aligned} f(x) &= 2x^3 + x^2 - 5x + 2 \\ &= 2(x + 2)(x - 1)(x - \tfrac{1}{2}) \\ &= (x + 2)(x - 1)(2x - 1) \end{aligned}$$

7. $f(3) = 18 - k = 9$, so $k = 9$.

8. If $f(x)$ is exactly divisible by $x - 3$, then $f(3) = 0$. But by substitution (direct or synthetic), $f(3) = 3k + 6$. Hence, $3k + 6 = 0$ and $k = -2$.

9. $f(-1) = -2a - 14 = 0$, so $a = -7$. Then $f(1) = -8$.

10. a) $(2x - 3)(x + 5)$

b) $(x - \frac{1 + \sqrt{5}}{2})(x - \frac{1 - \sqrt{5}}{2}) = \frac{1}{4}(2x - 1 - \sqrt{5})(2x - 1 + \sqrt{5})$

c) $(x + 2i)(x - 2i)$

d) $(x - 3 - 2i)(x - 3 + 2i)$

e) $x(x + \sqrt{5})(x - \sqrt{5})$

f) This answer is given in the text.

g) $9(x - \frac{-1 + 2i}{3})(x - \frac{-1 - 2i}{3}) = (3x + 1 - 2i)(3x + 1 + 2i)$

h) $2(x - \frac{2 + \sqrt{2}}{2})(x - \frac{2 - \sqrt{2}}{2}) = \frac{1}{2}(2x - 2 - \sqrt{2})(2x - 2 + \sqrt{2})$

It should be pointed out here that the factorization of polynomials into linear factors with real or imaginary coefficients is not unique. For example, the answer to part (a) given above is a particularly simple form (usually desirable), but other ways of factoring the given polynomial might include $2(x - \frac{3}{2})(x + 5)$, $\sqrt{2}(x - \frac{3}{2})(\sqrt{2}x + 5\sqrt{2})$, or even $\frac{1}{5}(2 + i)(x + 5)[(4 - 2i)x + (-6 + 3i)]$!

2-5. Locating Zeros of Polynomial Functions. Pages 58-65.

The statement of the Location Theorem is made plausible on the basis of a graphical interpretation. Since no formal study of continuity has been included, an intuitive appeal is the only justification for the assumption that the polynomial function is continuous in any closed interval.

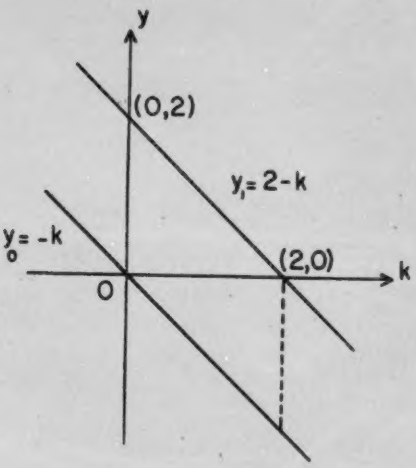
Use of the Location Theorem does not guarantee that we can find any of the real zeros of a polynomial function, but it does provide helpful clues in the search. The determination of upper and lower bounds for the zeros further limits the interval in which any real zeros are to be found.

Answers to Exercises 2-5. Page 66.

1. a) $-1 < x < 0$; $1 < x < 2$; and $2 < x < 3$
   b) $0 < x < 1$
   c) $1 < x < 2$
   d) $-2 < x < -1$; and two zeros such that $0 < x < 1$
   e) $-1 < x < 0$; $1 < x < 2$; and $2 < x < 3$
   f) $2 < x < 3$
   g) $-2 < x < -1$; $0 < x < 1$; $1 < x < 2$; and $5 < x < 6$

2. $f(x) = x^3 - 2x^2 + 3x - k$
   a) $f(0) = -k$ and $f(1) = 2 - k$
   For $-k$ and $2 - k$ to be different in sign, we must have $0 < k < 2$, since, if $k < 0$, both $-k > 0$ and $2 - k > 0$. If $k > 2$, both $-k < 0$ and $2 - k < 0$.

   Another way to see this is to graph $y_0 = -k$ and $y_1 = 2 - k$. A value of k for which y_0 and y_1 are both positive or both negative must be rejected. But when one y is above the k-axis and the other is below, we have a possible k-value. (See figure on next page.)



b) $f(1) = 2 - k$ and $f(2) = 6 - k$.
Hence, $2 < k < 6$.

2-6. Rational Zeros. Pages 66-71.

The student should be encouraged to make intelligent guesses for the zeros of a function. At this time he should use all of the resources at his command.

The relationships between the roots and the coefficients of polynomial equations, a topic commonly included in advanced algebra, are here introduced briefly for cubic equations in Exercises 14 to 18, rather than in the text. If time is short, these exercises may be omitted without loss of continuity. On the other hand, these relationships are suitable for further investigation by interested students, who should be encouraged to generalize the results for polynomial equations of degree $n > 0$.

For the benefit of those who may wish to see it, we spell out here in greater detail a part of the proof of Theorem 2-4 on rational zeros. In the proof in the text, Equation (3) states that

$$a_0q^n = pN,$$

where N is an integer. Hence, p divides a_0q^n a whole number of times; that is, N times. We wish to show that, since p and q have no common divisor greater than 1, it must be that p divides a_0.

First, we dispose of the special cases $p = 1$ or $q = 1$. The conclusion is obvious in either case; if $p = 1$, then of course p divides a_0; if $q = 1$, then Equation (3) becomes $a_0 = pN$, and obviously p divides a_0.

When neither p nor q is 1, we can appeal to the Fundamental Theorem of Arithmetic, that the factorization of positive integers is unique, to write p and q as products of prime factors:

$$p = \pm p_1p_2 \cdots p_k \ ; \qquad q = q_1q_2 \cdots q_m,$$

where $p_1, \ldots, p_k$ and $q_1, \ldots, q_m$ are positive primes. None of the p_j's are included among the q_i's and conversely, since p and q have no common divisor greater than 1. Now consider p and a_0q^n, with

$$a_0q^n = a_0\,(q_1{}^nq_2{}^n \cdots q_m{}^n) = \pm(p_1p_2\cdots p_k)N.$$

All of the p_i's must occur as divisors of $a_0(q_1{}^nq_2{}^n\cdots q_m{}^n)$. None of them occur among the $q_i{}^n$'s. Hence, all of them occur in the decomposition of a_0 into its prime factors. Therefore, p divides a_0.

Answers to Exercises 2-6. Pages 71-73.

1. a) -1/2, 2 b) -1/2, 0, 2
2. a) 1, 2, 3 b) 0, 1, 2, 3
3. a) No rational zeros b) 0
4. a) -1, 1/2, 1 b) -1, 0, 1/2, 1
5. -1/2, 3/2, 7/3
6. 4/3, $1 + \sqrt{2}$, $1 - \sqrt{2}$
7. No rational zeros
8. -2, -1, 2, 3
9. -2, 2 (Each of these is a zero of multiplicity two. See Section 2-8.)
10. -1, 1, 2, 3

11. $-3, -2, -1, 1, 2$

12. $-3, 5/3, 2 + \sqrt{3}, 2 - \sqrt{3}$

13. $x + \frac{1}{x} = n \Longleftrightarrow x^2 - nx + 1 = 0$ if $x \neq 0$. The discriminant for this quadratic is $n^2 - 4$. If $|n| < 2$, $n^2 < 4$ and $n^2 - 4 < 0$, which means that the roots are imaginary.

14. $(x + 2)(x - 1)(x - 3) = x^3 - 2x^2 - 5x + 6 = 0$

15. a) 2. It is the negative of the coefficient of x^2 in Exercise 14.

    b) -5. It is the same as the coefficient of x in Exercise 14.

    c) -6. It is the negative of the constant term in Exercise 14.

16. a) 3/2 b) -11/2 c) -3

    d) $x^3 - \frac{3}{2}x^2 - \frac{11}{2}x + 3 = 0$

    e) $(x + 2)(x - \frac{1}{2})(x - 3) = x^3 - \frac{3}{2}x^2 - \frac{11}{2}x + 3 = 0$

17. a) $(x - r_1)(x - r_2)(x - r_3)$

    $= x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3 = 0$

    b) $\frac{a_2}{a_3} = -(r_1 + r_2 + r_3)$

    $\frac{a_1}{a_3} = r_1r_2 + r_1r_3 + r_2r_3$

    $\frac{a_0}{a_3} = -r_1r_2r_3$

18. Any 3rd-degree polynomial function with zeros -1, 1, and 4 is

$$f\colon x \longrightarrow a_3(x^3 - 4x^2 - x + 4).$$

From this, $f(0) = 4a_3$. Since the exercise specifies that $f(0) = 12$, it follows that $4a_3 = 12$ and $a_3 = 3$. Hence, the required function is

$$f\colon x \longrightarrow 3x^3 - 12x^2 - 3x + 12.$$

2-7. Decimal Approximation of Irrational Zeros. Page 73.

One method for calculating approximate values of irrational zeros of polynomial functions is treated briefly in this section. Its virtue is that it requires only the techniques of synthetic substitution and linear (straight-line) interpolation, which are applied in the light of information obtained by the Location Theorem. We assume that students are familiar with the process of interpolation from previous courses, but a brief illustration here of the method may be helpful.

Suppose that for a given polynomial it is found that $f(2.3) = -0.62$ and $f(2.4) = 0.47$. Figure TC 2-7 shows a possible graphical interpretation. The curve intersects the x-axis at point P, and the value of x at this point is the desired zero of the function. (It is conceivable that the function might have three or some other odd number of zeros between 2.3 and 2.4. [See Figure 2-4a, page 60, in the text.] Geometrically, this means that the curve would cross the x-axis more than once in this interval. In such a situation, linear interpolation would probably not be helpful.) We assume that between points A and B the curve can be approximated by the straight line AB. This line intersects the x-axis at point Q, and the value of x at this point is a reasonable approximation of the zero at point P.

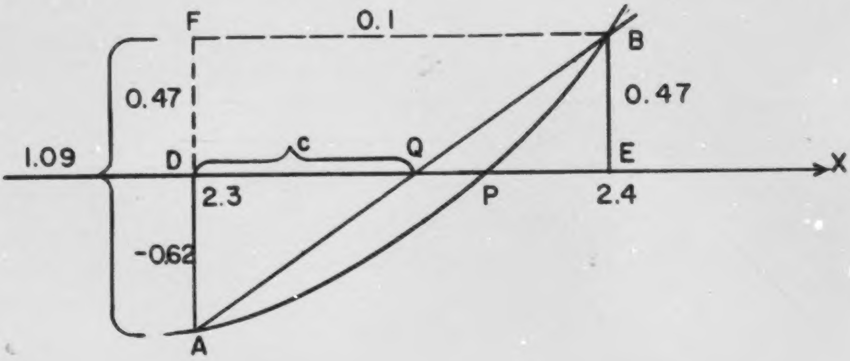


Figure TC 2-7. Linear Interpolation.

The value of x at point Q is $x = 2.3 + c$, where c is the length of the segment DQ. Since triangles ADQ and AFB are similar, it follows that $\frac{DQ}{FB} = \frac{AD}{AF}$. Substituting the lengths of the known segments (undirected distances), we obtain $\frac{c}{0.1} = \frac{0.62}{1.09}$, from which $c \approx 0.057$. Hence, an approximate value of x at point Q is $x \approx 2.3 + 0.06 = 2.36$.

Answers to Exercises 2-7. Pages 73-74.

1. $x \approx 3.0^+$, to the nearest 0.5.

2. a) $x \approx 2.0^+$, to the nearest 0.5.
   b) $x \approx 2.2^-$, to the nearest 0.1.

3. a) $x \approx 1.2^+$, to the nearest 0.1.
   b) $x \approx 1.21^+$, to the nearest 0.01.

4. $\sqrt[3]{20} \approx 2.71^+$, to the nearest 0.01.

2-8. Number of Zeros. Pages 74-80.

The statement of the General Form of the Fundamental Theorem of Algebra, that a polynomial function of degree $n > 0$ has at least one and at most n zeros, may come as a surprise to many teachers. Most of us are accustomed to saying that every polynomial of degree $n > 0$ has exactly n zeros. But from the point of view of set theory, the set of zeros of f_n is the solution set of the equation $f_n(x) = 0$. This set is $\{r_1, r_2, \ldots, r_k\}$ where $k \leq n$. It is possible for k to be less than n because in listing the elements of a set no element is repeated. For example, if $f_3: x \longrightarrow (x - 1)(x - 1)(x - 3)$, then the solution set $\{x: f_3(x) = 0\}$ is $\{1, 3\}$. In other words, the zeros of f_3 are 1 and 3. At the same time, however, we note that 1 is a zero of multiplicity two. Hence, the sum of the multiplicities of the zeros is exactly three, and this is also the degree of the given function.

Included in this section is the graph of a polynomial function that has zeros of multiplicity greater than one. We do not expect students to spend any effort at this time in drawing such graphs, but the question may arise in class as to what they look like. A more intensive treatment of this question appears in Chapter 3, Section 3-10, after more efficient means of sketching graphs have been developed.

Answers to Exercises 2-8. Pages 80-81.

1. (a) and (b) are unstable; (c), (d), and (e) are stable.
2. a) -1 of multiplicity two, and 2.
   3rd degree -- sum of multiplicities is 3.
   b) 1 of multiplicity two, and -2.
   3rd degree -- sum of multiplicities is 3.
   c) -1 of multiplicity three, and -2.
   4th degree -- sum of multiplicities is 4.
3. a) 1 of multiplicity two, -2 of multiplicity three.
   b) 1 of multiplicity three, -2 of multiplicity two.
   The solution set is $\{1, -2\}$ for both equations.
4. (a), (b), (c), (d) are not closed, as shown by the following list of exceptions:
   a) $2x - 1 = 0$ has a root not an integer.
   b) $x^2 - 2x - 2 = 0$ has roots which are not rational.
   c) $x^2 + 1 = 0$ has imaginary roots.
   d) $ix + i = 0$ has a real root.
   (e) is closed. (There is a theorem which establishes this fact.)
5. So far as the specific examples are concerned, $\sqrt[4]{-1}$ and $\sqrt[6]{-1}$ are solutions of $x^4 + 1 = 0$ and $x^6 + 1 = 0$, respectively. In each case, the Fundamental Theorem of Algebra guarantees the existence of a complex zero. If the student is familiar with De Moivre's Theorem, he will know how to obtain, respectively four and six complex-number solutions. More generally, any root of a complex number is a complex number. It is even the case that all complex powers (or roots) of complex numbers are complex numbers. Hence, "super-complex" numbers are unnecessary. (See Fehr, Howard F., Secondary Mathematics, A Functional Approach for Teachers, D. C. Heath, 1951.)

2-9. Complex Zeros. Pages 81-84.

In this section the treatment of the zeros of polynomial functions is extended to the case where the zeros are complex numbers (which may be real). The Complex-conjugates Theorem shows that if a polynomial* with real coefficients has complex zeros, with imaginary part not zero, then these zeros must occur in pairs (conjugates). An illustrative example shows that the coefficients need not be real if the complex (imaginary) zeros do not occur as conjugates. These ideas are extended in the exercises to include real zeros which are conjugate surds of the form $a + b\sqrt{c}$ and $a - b\sqrt{c}$ for specified values of c.

Answers to Exercises 2-9. Pages 84-85.

1. $[x - (2 + i)]\ [x - (2 - i)]\ [x - 1]\ [x - (3 - 2i)]\ [x - (3+2i)]$

$$= (x^2 - 4x + 5)\ (x - 1)\ (x^2 - 6x + 13)$$

$$= (x^4 - 10x^3 + 42x^2 - 82x + 65)\ (x - 1)$$

$$= x^5 - 11x^4 + 52x^3 - 124x^2 + 147x - 65$$

The coefficient of x^4 is -11. The sum of the zeros is $(2 + i) + (2 - i) + 1 + (3 - 2i) + (3 + 2i)$ which is 11. The sum is the negative of the coefficient of x^4. The constant term is -65. The product of the zeros is $(2 + i)\ (2 - i)\ (1)\ (3 - 2i)\ (3 + 2i)$ which is 65. The product is the negative of the constant term.

2. a) $x \to x - (2 + 3i) = x - 2 - 3i$

   b) $x \to [x - (2 + 3i)]\ [x - (2 - 3i)] =$
   $[(x - 2) - 3i]\ [(x - 2) + 3i] = x^2 - 4x + 13$

---

*Here the word "polynomial" is used loosely to avoid cumbersome sentence structure. More precisely, the statement might read, "... if a polynomial function, defined by a polynomial with real coefficients, has complex zeros with imaginary part not zero, then the zeros of the function must occur in pairs." Recall the statement on page 32 of this Commentary regarding the use of the word "polynomial".

3. a) $1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$

b) $-1, \frac{1 + i\sqrt{3}}{2}, \frac{1 - i\sqrt{3}}{2}$

c) $2, \frac{-1 + i\sqrt{15}}{2}, \frac{-1 - i\sqrt{15}}{2}$

d) $i, -i, 2i, -2i$

e) 1 of multiplicity two, $3i, -3i$

f) $-1, i, -i$, each of multiplicity two.

g) $1, i, -i$, each of multiplicity two. Note that these roots are the negatives of the roots in (f), as would be expected from inspection of the two equations.

4. 8

5. Using either of the proofs of Theorem 2-7 as a model, two proofs may be shown. Only one is given here.

Given: $f(a + b\sqrt{2}) = 0$, a and b are rational numbers, $b \neq 0$.

To prove: $f(a - b\sqrt{2}) = 0$

Proof: Let
$$\begin{aligned} p(x) &= [x - (a + b\sqrt{2})]\,[x - (a - b\sqrt{2})] \\ &= [(x - a) - b\sqrt{2}]\,[(x - a) + b\sqrt{2}] \\ &= (x - a)^2 - 2b^2. \end{aligned}$$

The coefficients of $p(x)$ are rational. If $f(x)$ is divided by $p(x)$ we get a quotient $q(x)$ and a remainder $r(x) = hx + k$, possibly of degree 1 (but no greater), where h, k, and all coefficients of $q(x)$ are rational. Thus,

$$f(x) = p(x) \cdot q(x) + hx + k.$$

This is an identity in x. By hypothesis, $f(a + b\sqrt{2}) = 0$, and from $p(x)$ above $p(a + b\sqrt{2}) = 0$, so we get

$$0 = 0 + ha + hb\sqrt{2} + k.$$

If hb is not zero, we get

$$\sqrt{2} = \frac{-ha - k}{hb},$$ where h, a, k, and b are rational, which is impossible. So $hb = 0$, and since $b \neq 0$, h must equal zero, and as a consequence k must equal zero. Therefore,

$$f(x) = p(x) \cdot q(x).$$

Since $p(a - b\sqrt{2}) = 0$, it follows that
$f(a - b\sqrt{2}) = 0$. q.e.d.

6. $x \rightarrow [x - (3 + 2\sqrt{2})]\,[x - (3 - 2\sqrt{2})] = x^2 - 6x + 1$

7. For $a + b\sqrt{3}$, the proof given in the answer to Exercise 5 will be correct if $\sqrt{3}$ is substituted for $\sqrt{2}$.

For $a + b\sqrt{4}$ there is no comparable theorem since the root is rational and there is no conjugate surd. If a proof like that in Exercise 5 is attempted, it breaks down at the step

$$\sqrt{2} = \frac{-ha - k}{hb}.$$

If 4 is substituted for 2, both sides are rational and the contradiction needed in the proof does not appear.

8. a) $f:\ x \rightarrow x^2 + (-2 + 2\sqrt{3})\,x - 3 + 2\sqrt{3}$

b) $f:\ x \rightarrow x^3 - 5x^2 - 9x - 3$

9. $f:\ x \rightarrow x^4 - 10x^2 + 1$ (The zeros of the function are $\sqrt{3} + \sqrt{2}$, $\sqrt{3} - \sqrt{2}$, $-\sqrt{3} + \sqrt{2}$, and $-\sqrt{3} - \sqrt{2}$.)

10. a) (1) Degree 2 $(x \rightarrow x^2 - 2\sqrt{2}\,x + 3)$

(2) Degree 2 $(x \rightarrow x^2 - 2x + 3)$

(3) Degree 2 $(x \rightarrow x^2 - 2\sqrt{2}\,x + 5)$

b) (1) Degree 4 $(x \rightarrow x^4 - 2x^2 + 9)$

(2) Degree 2 $(x \rightarrow x^2 - 2x + 3)$

(3) Degree 4 $(x \rightarrow x^4 + 2x^2 + 25)$

(Solution for part (1) of (a) and (b).

A polynomial function having only the zero $i + \sqrt{2}$ is $f:\ x \rightarrow x - (i + \sqrt{2})$. This function has imaginary coefficients, but we can obtain from it a function with real coefficients as follows. Write the equation $x - (i + \sqrt{2}) = 0$ in the form $x - \sqrt{2} = i$. Now square both members and rearrange the terms to obtain the equation

$$x^2 - 2\sqrt{2}\,x + 3 = 0.$$

This is the equation corresponding to the function given as the answer to part (1) of (a). It has real coefficients. If we now write this equation in the form $x^2 + 3 = 2\sqrt{2}\,x$ and again square both members, we obtain the equation $x^4 - 2x^2 + 9 = 0$ with rational coefficients. From this we have the answer to part (1) of (b).

An alternative procedure for the two parts of this question depends upon recognition of the fact that if $i + \sqrt{2}$ is a zero of a polynomial function with real coefficients, then the complex conjugate $-i + \sqrt{2}$ is also a zero of the function. Hence, the function of minimum degree having the zeros $i + \sqrt{2}$ and $-i + \sqrt{2}$ will be the answer to part (1) of (a). To obtain a function of minimum degree with rational coefficients, the additional zeros $i - \sqrt{2}$ and $-i - \sqrt{2}$ must be introduced. Hence, the function will be of 4th degree and will have the zeros $i + \sqrt{2}$, $-i + \sqrt{2}$, $i - \sqrt{2}$, and $-i - \sqrt{2}$. This is the function given as the answer to part (1) of (b).

Answers to Miscellaneous Exercises, Chapter 2. Pages 88-92.

This set of exercises contains problems of two kinds. Numbers 1 to 21 are review problems and are, for the most part, similar to those contained in the various sections of the chapter. Numbers 22 to 35 extend the ideas of the chapter somewhat and are in general more difficult than the first set.

1. a) $f(0) = 9$ b) $f(-3) = -84$ c) $f(\frac{1}{2}) = 7$

2. a) $f(0) = -3$ b) $f(-2) = -31$ c) $f(\frac{1}{3}) = -\frac{20}{9}$

3\.

	Quotient	Remainder
a)	$3x^3 + x^2 - 2x - 1$	-4
b)	$x^3 - x^2 + x - 1$	0
c)	$5x^2 + 4x - 2$	-16

4\.

	Quotient	Remainder
a)	$x^2 - x + 3$	4
b)	$27x^3 - 18x^2 + 12x - 8$	32

5. a)

1	-6	11	-6	
1	-8	27	-60	-2
1	-7	18	-24	-1
1	-6	11	-6	0
1	-5	6	0	1
1	-4	3	0	2
1	-3	2	0	3

$x^3 - 6x^2 + 11x - 6$
$= (x - 1)(x - 2)(x - 3)$

b)

1	-3	-4	12	
1	-5	6	0	-2
1	-4	0	12	-1
1	-3	-4	12	0
1	-2	-6	6	1
1	-1	-6	0	2
1	0	-4	0	3

$x^3 - 3x^2 - 4x + 12$
$= (x + 2)(x - 2)(x - 3)$

c)

1	-2	-1	2	0	
1	-4	7	-12	24	-2
1	-3	2	0	0	-1
1	-2	-1	2	0	0
1	-1	-2	0	0	1
1	0	-1	0	0	2
1	1	2	8	24	3

$x^4 - 2x^3 - x^2 + 2x$

$= x(x + 1)(x - 1)(x - 2)$

6. $f(2) = -2 + 8k = 22$; hence, $k = 3$

7. $f(-\frac{1}{3}) = \frac{7 - k}{3} = 0$; hence, $k = 7$.

8. a) $(2x + \sqrt{3})(2x - \sqrt{3}) = 4(x + \frac{\sqrt{3}}{2})(x - \frac{\sqrt{3}}{2}) = \ldots$

   b) $(\sqrt{7}x + 3i)(\sqrt{7}x - 3i)$

   c) $(x - 2 - i\sqrt{3})(x - 2 + i\sqrt{3})$

   d) $3(x - \frac{5 + \sqrt{37}}{6})(x - \frac{5 - \sqrt{37}}{6}) = \frac{1}{12}(6x - 5 - \sqrt{37})(6x - 5 + \sqrt{37})$

9. a) $-2 < x < -1, \quad 0 < x < 1, \quad 2 < x < 3$

   b) $1 < x < 2$

10. a) $-2, \quad 0, \quad \frac{3}{2}$

    b) $-2, -1, 1$

    c) $-2, \frac{3 + \sqrt{7}}{2}, \quad \frac{3 - \sqrt{7}}{2}$

    d) No rational roots. (By Theorem 2-4, if the equation had a rational root, it would have to be one of the following: ± 1, ± 2, $\pm 1/3$, $\pm 2/3$. By synthetic substitution we find that none of these is a root of the given equation.)

    e) $-3, 5/3, 2 + \sqrt{7}, \quad 2 - \sqrt{7}$

11. For any real number c, $g(c) = 2f(c)$. Hence, if both graphs are drawn with the same scales on the corresponding axes, each point on the graph of g will be twice as far from the x-axis as the corresponding point on the graph of f. Both graphs, however, will have the same x-intercepts; namely, -2, 1, and 3.

(These numbers are, of course, the zeros of each function.)

12. $f: x \longrightarrow (x + 1)(x - 2)(x - 3) = x^3 - 4x^2 + x + 6$

(Or use the relationships between the zeros and the coefficients.)

13. a) $f: x \longrightarrow -1(x^3 - 4x^2 + x + 6) = -x^3 + 4x^2 - x - 6$

b) $f: x \longrightarrow \frac{5}{2}(x^3 - 4x^2 + x + 6) = \frac{5}{2}x^3 - 10x^2 + \frac{5}{2}x + 15$

(See the discussion in this volume of Exercise 18, Section 2-6.)

14. A general procedure for problems of this kind is as follows: Any polynomial function of degree 3 can be written as $f_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.

From the given information,

$$f(2) = 8a_3 + 4a_2 + 2a_1 + a_0 = 0$$

$$f(3) = 27a_3 + 9a_2 + 3a_1 + a_0 = 0$$

$$f(0) = a_0 = 6$$

$$f(1) = a_3 + a_2 + a_1 + a_0 = 12$$

Solving these equations simultaneously, we obtain $a_3 = 5$, $a_2 = -24$, $a_1 = 25$, $a_0 = 6$.

Hence, the required function is:

$$f: x \longrightarrow 5x^3 - 24x^2 + 25x + 6$$

15. $x \approx 0.6^+$

16.

	Zeros	y-intercept	Behavior of Graph for large $\lvert x \rvert$
a)	2, multiplicity two	4	dominated by x^2
b)	2, multiplicity three	8	dominated by $-x^3$
c)	2, multiplicity four	48	dominated by $3x^4$
d)	1, multiplicity two -2, multiplicity one	-4	dominated by $-2x^3$

(For a discussion of the behavior of the graph of a polynomial function for large $\lvert x \rvert$, see the text, page 52. For example, the polynomial in part (a) of this Exercise may be written as $y = x^2 - 4x + 4$

$$= x^2(1 - \frac{4}{x} + \frac{4}{x^2}).$$

As $|x|$ increases, the value of the expression in parentheses comes closer and closer to 1. Hence, for large $|x|$ the x^2-term dominates all other terms, and the graph behaves like the graph of $y = x^2$. Note, however, that this graph does not actually approach $y = x^2$. Indeed, the vertical distance between this curve and that of $y = x^2$ is $|4x - 4|$, which can be made arbitrarily large by making $|x|$ large enough.

17. a) 1 of multiplicity two, -1 of multiplicity three
    b) -1 of multiplicity two, $-\frac{2}{3}$, and $-\frac{3}{2}$

18. a) -2 of multiplicity two, and 1
    b) -3 of multiplicity two, -2, and -1

Solution: $(x + 1)(x + 2)(x + 3) = (x+1)(x + 2)(x+3)(x+4)$

$$(x + 1)(x + 2)(x + 3) - (x + 1)(x + 2)(x + 3)(x + 4) = 0$$

$$(x + 1)(x + 2)(x + 3) [1 - (x + 4)] = 0$$

$$(x + 1)(x + 2)(x + 3)(-x - 3) = 0$$

$$-1(x + 1)(x + 2)(x + 3)^2 = 0$$

$x = -1$, -2, and -3 of multiplicity two.

19. a) 3 b) 3 c) 3

(Although the exercise does not ask for them, it may be helpful to give here the polynomials having the specified zeros.

a) $x^3 + (1 - i) x^2 + (-6 - i)x + 6i$

b) $x^3 - 4x^2 + (-48 + 14i)x + (192 - 56i)$

c) $x^3 + (2 - i)x^2 + (-3 - 4i)x + (-10 - 5i)$)

20. a) 4 b) 5 c) 4

(The polynomials are:

a) $x^4 + x^3 - 5x^2 + x - 6$

b) $x^5 - 4x^4 - 96x^3 + 384x^2 + 2500x - 10,000$

c) $x^4 - 6x^2 + 25$)

21. a) $f(x) = x^3 - 8x^2 + 19x - 14$

b) $f(x) = x^5 - x^4 + 14x^3 - 14x^2 + 121x - 121$

(The roots are 1, $\sqrt{2} + 3i$, $\sqrt{2} - 3i$, $-\sqrt{2} + 3i$, $-\sqrt{2} - 3i$)

22. a) $f(1) = -7 + i$
b) $f(i) = 0$
c) $f(1 + i) = -6 - 8i$
d) $f(1 - i) = -2 + 4i$
e) $f(-1 + i) = -6 + 8i$
f) $f(-1 - i) = -2 - 4i$

23. $f(x) = 2x^3 - 3x^2 + x - 5$
(See method of solution given for Exercise 14.)

24. a) Quotient $= (a - b)x - ab + b^2 = (x - a)(a - b)$
Remainder $= 0$

b) Quotient $= x^2 - (b + c)x + bc = (x - b)(x - c)$
Remainder $= 0$

25. Since $f_n(a) = 0$ by hypothesis, $f_n(x)$ is exactly divisible by $(x - a)$ or

$$f_n(x) = (x - a) \cdot q(x).$$

If now $x = b$,

$$f_n(b) = (b - a) \cdot q(b) \text{ which is equal to } 0.$$

Since $b \neq a$, $b - a \neq 0$, therefore, $q(b)$ must equal zero. If $q(b) = 0$, $q(x)$ is divisible by $(x - b)$ without a remainder, or

$$q(x) = (x - b) \cdot r(x), \text{ and so}$$
$$f_n(x) = (x - a) \cdot (x - b) \cdot r(x).$$

26. If $f_n(x) = x^n - a^n$ then $f_n(a) = a^n - a^n = 0$ which means $f_n(x)$ or $x^n - a^n$ is divisible by $(x - a)$. If n is even, $f_n(-a) = (-a)^n - a^n = (-1)^n(a^n) - a^n$. Since n is even, $(-1)^n = +1$ and $a^n - a^n = 0$.

27. Maximum number of:

	Positive Roots	Negative Roots
a)	2	1
b)	2	1
c)	1	1
d)	1	0
e)	0	1
f)	0	0

28. We give two proofs, the first one using the hint given in the text.

First proof:

To form a polynomial having $\sqrt{2} + \sqrt{3}$ as a root, we set $x - (\sqrt{2} + \sqrt{3}) = 0$. Now we have to rationalize the coefficients.

$x - \sqrt{2} = \sqrt{3}$ and squaring we get

$x^2 - 2\sqrt{2}x + 2 = 3.$

$x^2 - 1 = 2\sqrt{2}x$ and squaring again, we get

$x^4 - 2x^2 + 1 = 8x^2.$

$f(x) = x^4 - 10x^2 + 1 = 0.$

If $\sqrt{2} + \sqrt{3}$ is rational, it is a rational root of $f(x) = 0$. But if $\frac{p}{q}$ (in lowest terms) is a rational root of $f(x) = 0$, then by the rational roots theorem p must divide 1 and q must divide 1. Hence $\frac{p}{q} = +1$, or $\frac{p}{q} = -1$. But $\sqrt{2} + \sqrt{3} \neq 1$ and $\sqrt{2} + \sqrt{3} \neq -1$. Hence, $\sqrt{2} + \sqrt{3}$ is not a rational root of $f(x) = 0$ and therefore must be irrational.

Second proof: Assume that $\sqrt{3} + \sqrt{2}$ is rational; i.e., assume that $\sqrt{3} + \sqrt{2} = \frac{p}{q}$, where p and q are integers with no common integer divisor greater than 1, and $q \neq 0$.

$$\sqrt{3} + \sqrt{2} = \frac{p}{q},$$

or

$$\sqrt{3} = \frac{p}{q} - \sqrt{2}.$$

Squaring both members of this equation and then solving for $\sqrt{2}$, we obtain

$$\sqrt{2} = \frac{p^2 - q^2}{2pq} .$$

Since p and q are integers, this equation asserts that $\sqrt{2}$ is rational (or undefined if $p = 0$). But this conclusion contradicts the known fact that $\sqrt{2}$ is irrational. (The irrationality of $\sqrt{2}$ can be proved, if desired.)
Hence, the assumption that $\sqrt{2} + \sqrt{3}$ is rational is false. Therefore, $\sqrt{2} + \sqrt{3}$ is irrational.

29. a) $-2 + i$ b) $-2 - i$

30. a) $3 + 4i$

b) $(3 - 4i)(2 - i) = 2 - 11i$

c) $(3 + 4i)(2 - i) = 10 + 5i$

(The property referred to in the hint is this: if $f(x)$ is a polynomial with real coefficients and if $f(a + ib) = u + iv$, then $f(a - ib) = u - iv$. See the second proof of Theorem 2-7 given in the text.)

31. a) $\left.\begin{aligned} f(2 + \sqrt{3}) &= 6 + \sqrt{3} \\ f(2 - \sqrt{3}) &= 6 - \sqrt{3} \end{aligned}\right\}$ These numbers are conjugate surds.

b) $\left.\begin{aligned} g(2 + \sqrt{3}) &= 15 + 2\sqrt{3} \\ g(2 - \sqrt{3}) &= 9 - 6\sqrt{3} \end{aligned}\right\}$ The property illustrated in (a) does not hold.

32. a) $-3,\ 2$ c) 4
b) $-2,\ 2$ d) $-3 \le x < -2$ or
$2 \le x < 3$

33. The four terms of the sequence can be shown (by successive differences) to be of the form $n^2 + 1$. Hence, the polynomial of minimum degree is $n^2 + 1$. Now if we add to this any polynomial which vanishes at $n = 1, 2, 3,$ or 4, we will have another polynomial answering the examination question. The most obvious polynomial to add is $(n - 1)(n - 2)(n - 3)(n - 4)$, since this vanishes at $n = 1, 2, 3,$ or 4. If we multiply this by any nonzero polynomial A, the result will

be another polynomial satisfying the requirements. Since there is an unlimited number of polynomials of this form, a general answer is

$$n^2 + 1 + A(n - 1)(n - 2)(n - 3)(n - 4),$$

where A is any nonzero polynomial. (In particular, A could, of course, be any nonzero constant.)

34. If a, b, and c are zeros of $f: x \longrightarrow x^3 + 7x^2 + 5$, we are to find a polynomial function with zeros $a + 2$, $b + 2$, and $c + 2$. To do this, we want a function g having zeros of the form $x = r + 2$, where $r = a$, b, or c. From this, $r = x - 2$, and since r is any one of the zeros of f,

$$\begin{aligned} g: x \longrightarrow f(r) = f(x - 2) &= (x - 2)^3 + 7(x - 2)^2 + 5 \\ &= x^3 + x^2 - 16x + 25 \end{aligned}$$

is a polynomial function satisfying the requirements.

It should be noted that this method can be used only when every zero of the new function is related to a zero of the given function by the same rule; in this particular case by adding 2.

35. Using the method of Exercise 34, $x = 2r + 1$, where $r = a$, b, or c, gives us $r = \frac{x - 1}{2}$. Substituting this in the given polynomial, we get $(\frac{x - 1}{2})^3 + 7(\frac{x - 1}{2})^2 + 5$. This can be multiplied by a constant without affecting the zeros, so we expand and then multiply by 8 to eliminate fractions. This gives the required function,

$$g: x \longrightarrow x^3 + 11x^2 - 25x + 53.$$

Illustrative Test Questions for Chapter 2

Teachers should refer to page 26 of this commentary for remarks about the use of these illustrative test questions. In this particular set of questions, the items marked with an asterisk are not necessarily more difficult, but deal with ideas that have not been emphasized in the text. For example, problems like Exercises 13 and 16 are discussed in the text only in the set of Miscellaneous Exercises at the end of Chapter 2.

1. If $f(x) = x^3 - 6x^2 + 11x + 10$, find $f(1)$, $f(2)$, and $f(3)$.

2. Given $f(x) = 2x^2 - 3x - 2$, find the points on the graph of $y = f(x)$ where the graph crosses

   a) the x-axis
   b) the y-axis
   c) the line $x = 7$
   d) the line $y = 7$

3. Given $f : x \rightarrow x^5 - 5x^3 + 5x^2 - 1$.
   a) Find all zeros of f.
   b) Write the polynomial in factored form as a product of linear factors with real coefficients.

4. Find all zeros of $f : x \rightarrow 3x^5 + 4x^4 - 26x^3 - 42x^2 + 7x + 6$.

5. Find the real root of $x^3 - 3x + 1 = 0$ between 0 and 1, correct to 2 decimal places.

6. Find the quotient and remainder when $3x^4 + 7x^3 - 5x^2 - x + 2$ is divided by $x + 3$. Check your answer by multiplication.

. Form an equation of minimum degree having the roots -2, 2, and $3 + i$, if
   a) imaginary coefficients are allowed;
   b) the coefficients must be real.

*8. Form an equation of minimum degree having the roots 2 and $5 - 3\sqrt{2}$, if
   a) irrational coefficients are allowed;
   b) the coefficients must be rational.

*9. Find the quotient and remainder when $2x^3 + (2 - i)x + 3$ is divided by $x + 1$.

*10. Find a polynomial $f(x)$ for which $f(0) = 8$, if it is known that f has the zeros -1, 1, -2, and 2.

*11. Which of the following are factors of

$$f(x) = 2x^{73} - 4x^{56} + x^{37} - 3x + 4\ ?$$

a) $x - 2$

b) $x - 1$

c) $x + 1$

d) $x + 2$

*12. Given the equation

$$(x + 1)(x + 2)^2(x + 3) = (x + 1)(x + 2)(3x + 5),$$

find the roots and their multiplicities.

*13. If $f(x)$ and $g(x)$ are polynomials with real coefficients such that $f(2 + 3i) = 1 - i$ and $g(2 + 3i) = -3 + 4i$, evaluate $f(2 - 3i) \cdot g(2 - 3i)$.

*14. Find an equation of minimum degree with rational coefficients having $2 + \sqrt{3}$ and $2 + i\sqrt{3}$ as roots.

*15. Given the equation $3x^3 - 6x^2 + x + 2 = 0$, find, without solving the equation,

a) the sum of its roots,

b) the product of its roots.

*16. Without solving, find the maximum possible number of positive and negative roots of the equation $x^5 + 3x^3 + 4x^2 - 4 = 0$.

Answers to Illustrative Test Questions for Chapter 2.

1. $f(1) = 16$ $\quad f(2) = 16$ $\quad f(3) = 16$

2. a) $(-1/2, 0)$ and $(2, 0)$, since $2x^2 - 3x - 2 = 0$ when $x = -1/2$ or 2.

b) $(0, -2)$, since $f(0) = -2$.

c) $(7, 75)$, since $f(7) = 75$.

d) $(-3/2, 7)$ and $(3, 7)$, since $2x^2 - 3x - 2 = 7$ when $x = -3/2$ or 3.

3. a) The only possible rational zeros are 1 and -1.

1	0	-5	5	0	-1	
1	1	-4	1	1	0	1
1	2	-2	-1	0	1	
1	3	1	0	1		

$x^2 + 3x + 1 = 0$ when $x = \frac{-3 \pm \sqrt{5}}{2}$

Hence, the zeros are 1 of multiplicity three, $\frac{-3 + \sqrt{5}}{2}$, and $\frac{-3 - \sqrt{5}}{2}$.

b) $f(x) = (x - 1)^3\left(x - \frac{-3 + \sqrt{5}}{2}\right)\left(x - \frac{-3 - \sqrt{5}}{2}\right)$

$= \frac{1}{4}(x - 1)^3(2x + 3 - \sqrt{5})(2x + 3 + \sqrt{5})$

4. The possible rational zeros are ± 1, ± 2, ± 3, ± 6, $\pm 1/3$, and $\pm 2/3$.

3	4	-26	-42	7	6	
3	7	-19	-61	-54	-48	1
3	10	-6	-54	-101	-196	2
3	13	13	-3	-2	0	3 ✓
3	10	3	-6	4	-1	
3	7	-1	-1	0	-2 ✓	
3	8	5/3	-4/9	1/3		
3	9	5	7/3	2/3		
3	6	-3	0	-1/3 ✓		

$3x^2 + 6x - 3 = 3(x^2 + 2x - 1) = 0$ when $x = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}$

The zeros are -2, $-1/3$, 3, $-1 + \sqrt{2}$, and $-1 - \sqrt{2}$.

5. $f(x) = x^3 - 3x + 1 = 0$

1	0	-3	1	
1	1	-2	-1	1
1	.5	-2.75	-.375	.5
1	.4	-2.84	-.136	.4 ←
1	.3	-2.91	.127	.3

Therefore, $x = 0.3 + c$ By interpolation, $\frac{c}{.1} = \frac{.127}{.263}$

$x \approx 0.35$ $c \approx 0.048$

6. Quotient $= 3x^3 - 2x^2 + x - 4$, remainder $= 14$

Check: $(x + 3)(3x^3 - 2x^2 + x - 4) = 3x^4 + 7x^3 - 5x^2 - x - 12$

Adding the remainder 14 gives the original polynomial.

7. a) $(x + 2)(x - 2)(x - 3 - i) = x^3 + (-3 - i)x^2 - 4x + (12 + 4i) = 0.$

b) $(x + 2)(x - 2)(x - 3 - i)(x - 3 + i) = (x^2 - 4)(x^2 - 6x + 10)$

$= x^4 - 6x^3 + 6x^2 + 24x - 40 = 0$

*8. a) $(x - 2)(x - 5 + 3\sqrt{2}) = x^2 + (-7 + 3\sqrt{2})x + (10 - 6\sqrt{2}) = 0$

b) $(x - 2)(x - 5 + 3\sqrt{2})(x - 5 - 3\sqrt{2}) = (x - 2)(x^2 - 10x + 7)$

$= x^3 - 12x^2 + 27x - 14 = 0$

*9.

2	0	2-i	3	-1
	-2	2	-4+i	
2	-2	4-i	-1+i	

Quotient $= 2x^2 - 2x + 4 - i$, remainder $= -1 + i$

*10. $f(x) = a_4(x + 1)(x - 1)(x + 2)(x - 2)$

$= a_4(x^2 - 1)(x^2 - 4) = a_4(x^4 - 5x^2 + 4)$

Since $f(0) = 4a_4 = 8$, $a_4 = 2$

Hence, $f(x) = 2x^4 - 10x^2 + 8$

*11. (b) and (c)

Since $f(1) = f(-1) = 0$, $(x - 1)$ and $(x + 1)$ are factors. It should be obvious that the size of the exponents makes values of 2 and -2 impossible. $2(\pm 2)^{73}$ contains 23 digits, while $-4(\pm 2)^{56}$ contains only 18 digits. Hence, the first term will be numerically approximately 100,000 times as large as the second term, and $f(\pm 2)$ could not possibly be 0.

*12. If $(x + 1)(x + 2)^2(x + 3) = (x + 1)(x + 2)(3x + 5)$, then $(x + 1)(x + 2)\,[\,(x + 2)(x + 3) - (3x + 5)\,] = 0$.

$$(x + 1)(x + 2)(x^2 + 5x + 6 - 3x - 5) = 0$$

$$(x + 1)(x + 2)(x^2 + 2x + 1) = 0$$

$$(x + 1)^3(x + 2) = 0$$

The roots are -1 of multiplicity three, and -2.

*13. $f(2 - 3i) \cdot g(2 - 3i) = (1 + i)(-3 - 4i) = 1 - 7i$

(See Exercise 30 in the Miscellaneous Exercises for Chapter 2.)

*14. $(x - 2 - \sqrt{3})(x - 2 + \sqrt{3})(x - 2 - i\sqrt{3})(x - 2 + i\sqrt{3})$

$= (x^2 - 4x + 1)(x^2 - 4x + 7) = x^4 - 8x^3 + 24x^2 - 32x + 7 = 0$

*15. a) The sum of the roots $= -a_2/a_3 = +2$

b) The product of the roots $= -a_0/a_3 = -2/3$

(See Exercises 15 to 18 in Section 2-6.)

*16. $f(x)$ has one variation in sign, so there is a maximum of 1 positive root.

$f(-x)$ has two variations in sign, so there is a maximum of 2 negative roots.

(See Exercise 27 in the Miscellaneous Exercises for Chapter 2.)

Chapter 3

TANGENTS TO GRAPHS OF POLYNOMIAL FUNCTIONS

Introduction.

Our treatment of the tangents to polynomial graphs is based upon (a) a factoring procedure and (b) the solution of a simple inequality involving absolute values. We illustrate this statement by an example.

To study the graph G of $f: x \longrightarrow 2 + 3x - 4x^2$ near $P(0, 2)$, we first write

$$2 + 3x - 4x^2 = 2 + (3 - 4x)x \quad (1) \text{ (Factoring step)}$$

This step may be motivated by observing that we want a straight line through $(0, 2)$, hence a line whose equation has the form

$$y = 2 + mx \quad (2)$$

where m is the (constant) slope. We therefore write (1) as nearly as possible in the form (2). Note that in this chapter it is convenient to write polynomials in ascending powers of x.

We cannot hope, of course, to make (1) exactly like (2) since $3 - 4x$ is not a constant. Our problem is to replace $3 - 4x$ by the right constant m to obtain the equation of the tangent.

It is easy to guess that the constant m which we seek is simply 3. The intuitive content of this guess is that $-4x$ is arbitrarily small for x numerically small enough. Since $-4x$ can be either positive or negative (depending on the sign of x), it is advantageous to use absolute value notation and write our statement as follows:

$|-4x|$ is arbitrarily small for $|x|$ small enough.

To formalize this statement, we translate "is arbitrarily small" by "$<\epsilon$, where ϵ is any positive number, however small."

We then have

$|-4x| < \epsilon$ for $|x|$ small enough.

Since the absolute value of a product is the product of the absolute values (See Section 1-4) we may write

$$4|x| < \epsilon \quad \text{for} \quad |x| \quad \text{small enough.} \quad (3)$$

Since $4|x| < \epsilon$ is equivalent to $|x| < \frac{\epsilon}{4}$, (3) may be replaced by the more precise statement

$$4|x| < \epsilon \quad \text{for} \quad |x| < \frac{\epsilon}{4} .$$

That is, we now know exactly what values of $|x|$ are small enough.

In the text, a method has been given for visualizing the meaning of this result in terms of the graph. We may refer to this picture as the "wedge" interpretation. For the present example, we are assured that if we stay within the vertical strip $|x| < \frac{\epsilon}{4}$ (See Figure TC 3-1), the graph lies in the wedge bounded by the two lines

L_1: $y_1 = 2 + (3 + \epsilon)x$.

and L_2: $y_2 = 2 + (3 - \epsilon)x$.

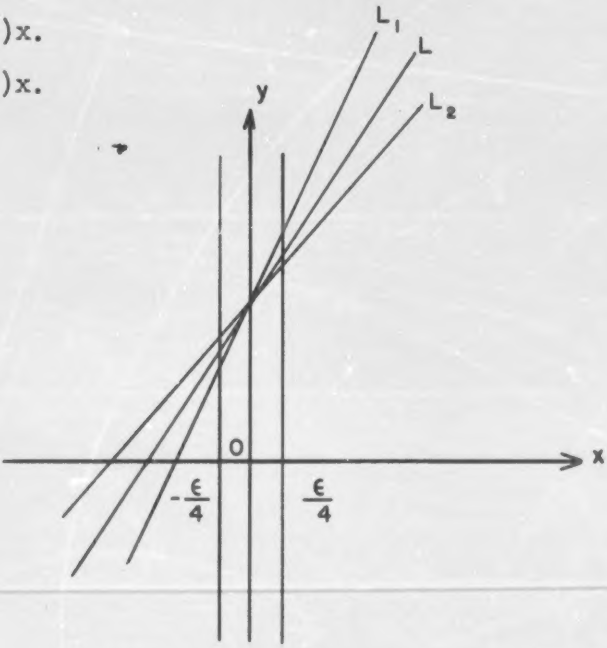


Figure TC 3-1

Of course L: $y = 2 + 3x$ is the only straight line which lies in all of these wedges and therefore the only line which approximates G in all intervals about P.

It is apparent that G, in fact, lies below L on both sides of P. If you wish you may introduce the phrase "G is concave downward at P" to describe this "shape." This terminology has not been used in the text of the chapter.

The modifications of this discussion for the case where G is concave upward or has a point of inflection at P will be clear from the examples in the text.

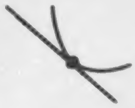
Teachers will note that we have used neither calculus notation nor calculus language. In fact, we have not used (and the teacher is urged not to use) even the traditional concept of a secant PQ and its associated slope. Consequently, we have not defined the slope of the tangent to be the limit of the slope ($\frac{\text{rise}}{\text{run}}$) of the secant as the run approaches zero. Indeed the words "limit" and "approaches" are not used. (The customary symbol "⟶" for "approaches" occurs in the text only as part of the mapping notation for a function and, of course, has a different meaning.)

The choice of the present mode of exposition was deliberate. The idea behind the concept of limit has here been reduced to its simplest core. This treatment, therefore, furnishes an excellent preparation for a full course in calculus. We believe, moreover, that in the time available for the study of tangents, it would be difficult to teach the conventional method without incurring the serious risks of misunderstanding which historically have been associated with it. The method adopted here is simple and logically sound and it does not raise the difficulties connected with the limits of quotients.

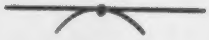
In Chapters 4 and 5, the wedge method will be applied to find the slopes of the graphs of exponential functions and of circular functions.

Answers to Exercises 3-2. Page 97.

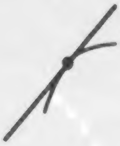
1. $y = 1 - x$



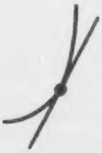
2. $y = 4$



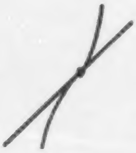
3. $y = 2 + 3x$



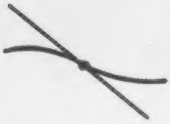
4. $y = 3 + 2x$



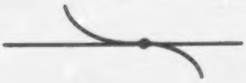
5. $y = 1 + x$



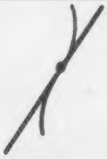
6. $y = 1 - x$



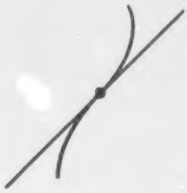
7. $y = 2$



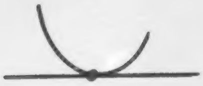
8. $y = 1 + 2x$



9. $y = x$



10. $y = 0$



Answers to Exercises 3-3. Pages 100-101.

1. Since $-.01 < x < .01$

$$.99 < 1 + x < 1.01.$$

If $x > 0$, we have

$$.99x < (1 + x)x < 1.01x$$

and $1 + .99x < 1 + (1 + x)x < 1 + 1.01x.$

If $x < 0$,

$$.99x > (1 + x)x > 1.01x$$

and $1 + .99x > f(x) > 1 + 1.01x.$

2. a) For $0 < x < .01$

$$1 < 1 + x < 1.01$$

$$1 + x < 1 + (1 + x)x < 1 + 1.01x$$

b) For $-.01 < x < 0$

$$.99 < 1 + x < 1$$

$$.99x > (1 + x)x > x$$

$$1 + .99x > f(x) > 1 + x.$$

L_1: $y = 1 + 1.01x$

L : $y = 1 + x$

L_2: $y = 1 + .99x$

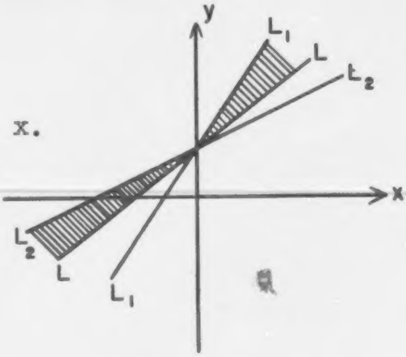


Figure for Ex. 2.

3. $x^2 > 0$ for all $x \neq 0$.

   Hence $f(x) > 1 + x, \quad x \neq 0$.

4. If $0 < x < .1$, then $0 < x^2 < .01$, $1 < 1 + x^2 < 1.01$ and $x < (1 + x^2)x = f(x) < 1.01x$.
   If $0 > x > -.1$, then $0 < x^2 < .01$, $1 < 1 + x^2 < 1.01$, and $x > (1 + x^2)x = f(x) > 1.01x$.

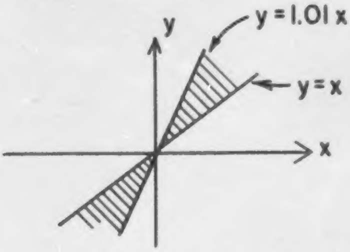


Figure for Ex. 4

5. a) (0, 2)

   b) $|x| < .01 \Longleftrightarrow -.01 < x < .01 \Longleftrightarrow -.01 < -x < .01 \Longleftrightarrow$
   $2.99 < 3 - x < 3.01$. (1)
   If $x > 0$, (1) is equivalent to $2.99x < (3 - x)x < 3.01x$
   or $2 + 2.99x < 2 + (3 - x)x = f(x) < 2 + 3.01x$, and
   if $x < 0$ the inequalities are reversed.

   c)

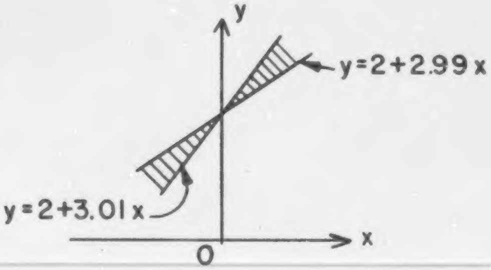


Figure for Ex. 5

6. If $x \neq 0$, $x^2 > 0$, and hence
$2 + 3x - x^2 < 2 + 3x$

L_1: $y = 2 + 3.01x$

L: $y = 2 + 3x$

L_2: $y = 2 + 2.99x$

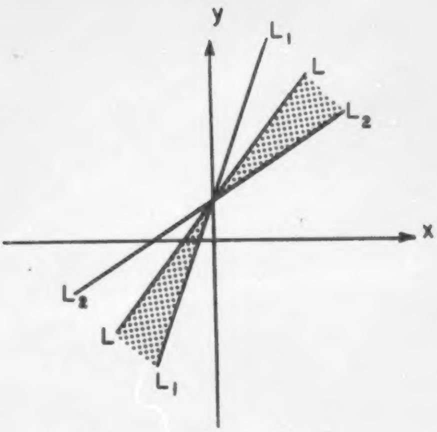


Figure for Ex. 6

7. a) $x^2 - 2x - 1 = -1 + (-2 + x)x$; f: $x \longrightarrow -1 + (-2 + x)x$.

b) If $0 < x < .01$, the lower bound is easily improved thus
$-2 < -2 + x < -1.99$
$-2x < (-2 + x)x < -1.99x$
$-1 - 2x < f(x) < -1 - 1.99x$

c) L_1: $-1 - 2.01x = y$
L: $y = -1 - 2x$
L_2: $y = -1 - 1.99x$

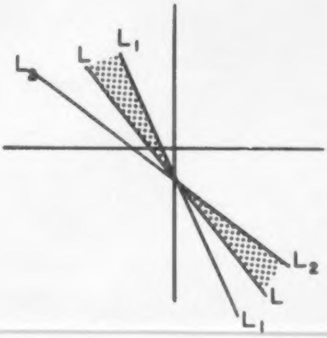


Figure for Ex. 7

8. a) f: $x \longrightarrow 3 - 5x - 4x^2 = 3 + (-5 - 4x)x$

b) If $|-4x| < \epsilon$, i.e. $|x| < \frac{\epsilon}{4}$,

f(x) lies between $y = 3 + (-5 + \epsilon)x$
and $y = 3 + (-5 - \epsilon)x$.

Thus if $|x| < .02$, $\epsilon = .08$
and $f(x)$ lies between $y = 3 - 4.92x$
and $y = 3 - 5.08x$
near $P(0, 3)$.

c) If $\epsilon = .002$, and $|-4x| < .002$, $|x| < .0005$.

3-4. The Behavior of the Graph Near P. (continued) Pages 101-104.

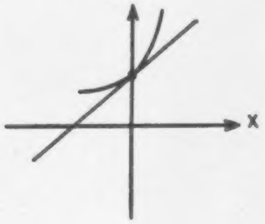
The notion of dominance is basic to an understanding of the development in this section. For this reason a review and extension of Section 2-3, page 52, is suggested. We there observed that for $|x|$ very large, the term of highest degree of the polynomial dominates the lower degree terms. In the present section we are concerned with the situation when $|x|$ is very small. In this case, the term of lowest degree dominates the higher degree terms. Thus if

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

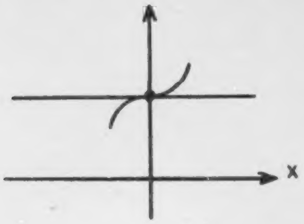
where $a_n \neq 0$, the first term $a_n x^n$ dominates all other terms if $|x|$ is sufficiently large; the term of lowest degree, a_0, dominates all other terms for $|x|$ sufficiently small. Furthermore, we note that the term $a_1 x$ dominates all terms of higher degree for $|x|$ sufficiently small. This follows at once since we can write $g(x) = f(x) - a_0$ and use the fact that the term of lowest degree, $a_1 x$, dominates the other terms of $g(x)$ for $|x|$ sufficiently small.

Answers to Exercises 3-4. Page 105.

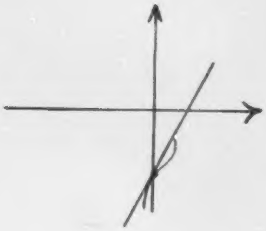
1. $f(0) = 2$, tangent at $(0, 2)$ is $y = 2 + x$.



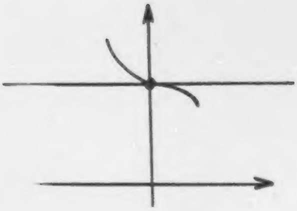
2. Point of inflection at (0, 2); equation of tangent is $y = 2$.



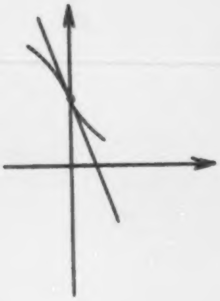
3. $P(0, -1)$; tangent: $y = -1 + 2x$.



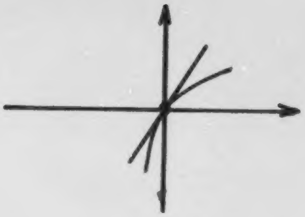
4. $P(0, 4)$; tangent: $y = 4$.



5. $P(0, 4)$; tangent: $y = 4 - 3x$.



6. $P(0, 0)$ tangent: $y = 2x$.



7. $f(x) = 2 + x + (3 - x)x^2$

If $|x| < \epsilon$, $f(x)$ lies between $2 + x + (3 + \epsilon)x^2$ and $2 + x + (3 - \epsilon)x^2$.

For $\epsilon = .01$, $|x| < .01$.

8. $f(x) = 2 + x^3 - x^4 = 2 + (1 - x)x^3$

If $|x| < \epsilon$, $f(x)$ lies between $2 + (1 + \epsilon)x^3$ and $2 + (1 - \epsilon)x^3$.

For $\epsilon = .01$, $|x| < .01$.

9. $f(x) = 1 + 2x - x^2 + 4x^3 = 1 + 2x + (-1 + 4x)x^2$.

If $|4x| < \epsilon$, or $|x| < \frac{\epsilon}{4}$, $f(x)$ lies between $1 + 2x + (-1 + \epsilon)x^2$ and $1 + 2x + (-1 - \epsilon)x^2$. If $\epsilon = .01$, $|x| < .0025$.

10. $f(x) = 4 - 2x^3 + x^4 = 4 + (-2 + x)x^3$.

If $|x| < \epsilon$, $f(x)$ lies between $4 + (-2 + \epsilon)x^3$ and $4 + (-2 - \epsilon)x^3$. If $\epsilon = .01$, $|x| < .01$.

11. $f(x) = 4 - 3x + x^3 - 7x^5 = 4 - 3x + (1 - 7x^2)x^3$.

If $|7x^2| < \epsilon$, or $x^2 < \frac{\epsilon}{7}$, $f(x)$ lies between $4 - 3x + (1 + \epsilon)x^3$ and $4 - 3x + (1 - \epsilon)x^3$.

If $\epsilon = .01$, $|x| < \sqrt{\frac{1}{700}} = \frac{1}{70}\sqrt{7} \approx 0.038$

12. $f(x) = 2x - x^2 + 4x^3 = 2x + (-1 + 4x)x^2$.

If $|x| < \frac{\epsilon}{4}$, $f(x)$ lies between $2 + (-1 + \epsilon)x^2$ and $2 + (-1 - \epsilon)x^2$. If $\epsilon = .01$, $|x| < .0025$.

3-5. The Tangent to the Graph at an Arbitrary Point P and the Shape Near P.

Pages 105-109.

Let us consider the problem of this section from an intuitive geometrical standpoint. Although this treatment is not included in the text, it may be considered as optional or supplementary.

The problem of finding the tangent to a polynomial graph G at an arbitrary point $P\big(h, f(h)\big)$ quickly reduces to the problem of translating the y-axis so that it passes through point P. Thereafter, procedures previously developed will enable us to solve the problem.

To illustrate we consider Example 1 on page 106.

Example 1. Find the tangent to the graph G of $f: x \longrightarrow 4 - 3x + 2x^2$ at $P(1, 3)$.

We translate the y-axis so that it passes through $P(1, 3)$, as shown by the dotted line in Figure TC 3-5. Since the y-axis has been shifted one unit to the right, any point on the xy-plane has coordinates $\big((x - 1), y\big)$ with respect to the translated axes; more specifically, any point on G has the new coordinates $\big((x - 1), f(x)\big)$. If we write $(x - 1) = x'$, then the expansion of $f(x)$ in powers of $(x - 1)$ becomes $f(x) = 3 + x' + 2x'^2$. By the method of Section 3-4 we write the equation of the tangent

$$y = 3 + x'$$

or

$$y = 3 + (x - 1)$$

at point P which has coordinates $(0, 3)$ with respect to the translated y-axis.

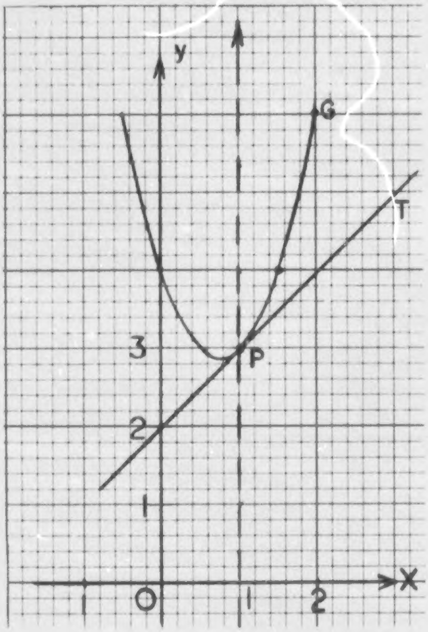


Figure TC 3-5

The teacher should note that some of the exercises in this section may be reserved for practice material or review at the close of the chapter. For example, an appropriate assignment might be Exercises 1a, 1b, 1d, 2a, 2b, 3a, 3b, 3c.

Answers to Exercises 3-5. Pages 109-110.

1. a)

$$\begin{array}{lllll} 1 & 2 & 4 & 3 & \lfloor\underline{2} \\ 1 & 4 & 12 & 27 & \\ 1 & 6 & 24 & & \\ 1 & 8 & & & \end{array}$$

$$f(x) = (x - 2)^3 + 8(x - 2)^2 + 24(x - 2) + 27$$

$$T:\ y = 24(x - 2) + 27 = 24x - 21$$

b)

$$\begin{array}{lllll} 2 & 4 & 0 & 3 & \lfloor\underline{-3} \\ 2 & -2 & 6 & -15 & \\ 2 & -8 & 30 & & \\ 2 & -14 & & & \end{array}$$

$$f(x) = 2(x + 3)^3 - 14(x + 3)^2 + 30(x + 3) - 15$$

$$T:\ y = 30(x + 3) - 15 = 30x + 75$$

c)

$$\begin{array}{lllll} 4 & -3 & 2 & 1 & \lfloor\underline{-4} \\ 4 & -19 & 78 & -311 & \\ 4 & -35 & 218 & & \\ 4 & -51 & & & \end{array}$$

$$f(x) = 4(x + 4)^3 - 51(x + 4)^2 + 218(x + 4) - 311$$

$$T:\ y = 218(x + 4) - 311 = 218x + 561$$

d)

$$\begin{array}{llllll} 5 & 0 & -3 & 2 & 1 & \lfloor\underline{\tfrac{1}{2}} \\ 5 & \frac{5}{2} & -\frac{7}{4} & \frac{9}{8} & \frac{25}{16} & \\ 5 & 5 & \frac{3}{4} & \frac{3}{2} & & \\ 5 & \frac{15}{2} & \frac{9}{2} & & & \\ 5 & 10 & & & & \end{array}$$

$$f(x) = 5(x - \tfrac{1}{2})^4 + 10(x - \tfrac{1}{2})^3 + \tfrac{9}{2}(x - \tfrac{1}{2})^2 + \tfrac{3}{2}(x - \tfrac{1}{2}) + \tfrac{25}{16}$$

$$T:\ y = \tfrac{3}{2}(x - \tfrac{1}{2}) + \tfrac{25}{16} = \tfrac{3}{2}x + \tfrac{13}{16}$$

e)
$$\begin{array}{rrrrl} 4 & 1 & 3 & 0 & \big|\underline{\;3\;} \\ 4 & 13 & 42 & 126 & \\ 4 & 25 & 117 & & \\ 4 & 37 & & & \end{array}$$

$f(x) = 4(x - 3)^3 + 37(x - 3)^2 + 117(x - 3) + 126$

T: $y = 117(x - 3) + 126 = 117x - 225$

f)
$$\begin{array}{rrrrl} 2 & 1 & -16 & -24 & \big|\underline{\;-2\;} \\ 2 & -3 & -10 & -4 & \\ 2 & -7 & 4 & & \\ 2 & -11 & & & \end{array}$$

$f(x) = 2(x + 2)^3 - 11(x + 2)^2 + 4(x + 2) - 4$

T: $y = 4(x + 2) - 4 = 4x + 4$

2. a)
$$\begin{array}{rrrrl} 3 & -5 & 2 & 1 & \big|\underline{\;-1\;} \\ 3 & -8 & 10 & -9 & \\ 3 & -11 & 21 & & \\ 3 & -14 & & & \end{array}$$

$f(x) = 3(x + 1)^3 - 14(x + 1)^2 + 21(x + 1) - 9$

b)
$$\begin{array}{rrrrl} 2 & 0 & -5 & 0 & \big|\underline{\;2\;} \\ 2 & 4 & 3 & 6 & \\ 2 & 8 & 19 & & \\ 2 & 12 & & & \end{array}$$

$f(x) = 2(x - 2)^3 + 12(x - 2)^2 + 19(x - 2) + 6$

c)
$$\begin{array}{rrrrl} 1 & -7 & 3 & 4 & \big|\underline{\;2\;} \\ 1 & -5 & -7 & -10 & \\ 1 & -3 & -13 & & \\ 1 & -1 & & & \end{array}$$

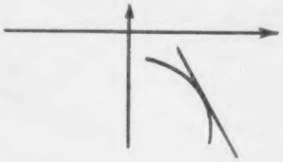
$f(x) = (x - 2)^3 - (x - 2)^2 - 13(x - 2) - 10$

d) $$\begin{array}{rrrrl} 1 & -2 & 1 & -1 & \underline{|\,-\frac{1}{2}} \\ 1 & -\frac{5}{2} & \frac{9}{4} & -\frac{17}{8} & \\ 1 & -3 & \frac{15}{4} & & \\ 1 & -\frac{7}{2} & & & \end{array}$$

$$f(x) = (x + \tfrac{1}{2})^3 - \tfrac{7}{2}(x + \tfrac{1}{2})^2 + \tfrac{15}{4}(x + \tfrac{1}{2}) - \tfrac{17}{8}$$

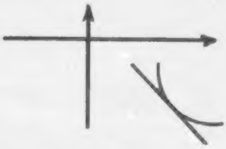
3. a) $$\begin{array}{rrrrl} 1 & -7 & 3 & 4 & \underline{|\,2} \\ 1 & -5 & -7 & -10 & \\ 1 & -3 & -13 & & \\ 1 & -1 & & & \end{array}$$

T: $y = -13(x-2) - 10 = -13x + 16$



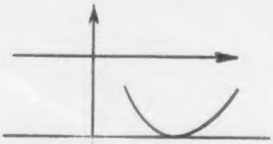
b) $$\begin{array}{rrrrl} 1 & -6 & 6 & -1 & \underline{|\,3} \\ 1 & -3 & -3 & -10 & \\ 1 & 0 & -3 & & \\ 1 & 3 & & & \end{array}$$

T: $y = -3(x-3) - 10 = -3x - 1$



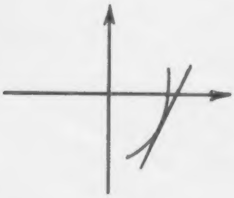
c) $$\begin{array}{rrrrrl} 3 & -4 & 0 & 0 & 0 & \underline{|\,1} \\ 3 & -1 & -1 & -1 & -1 & \\ 3 & 2 & 1 & 0 & & \\ 3 & 5 & 6 & & & \end{array}$$

T: $y = -1$



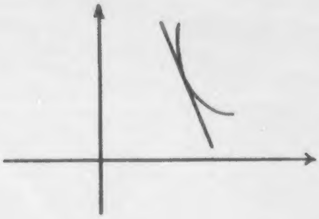
d) $$\begin{array}{rrrrl} 2 & -4 & -5 & 9 & \underline{|\,2} \\ 2 & 0 & -5 & -1 & \\ 2 & 4 & 3 & & \\ 2 & 8 & & & \end{array}$$

T: $g(t) = 3(t - 2) - 1 = 3t - 7$



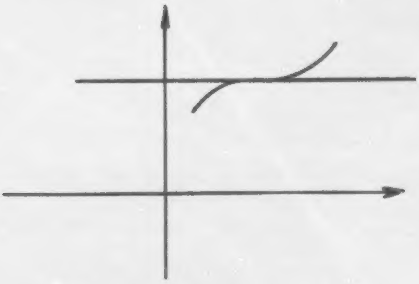
e) $$\begin{array}{rrrrl} 2 & -3 & -12 & 14 & \lfloor\underline{1} \\ 2 & -1 & -13 & 1 & \\ 2 & 1 & -12 & & \\ 2 & 3 & & & \end{array}$$

T: $y = -12(x-1) + 1 = -12x + 13$



f) $$\begin{array}{rrrrl} 2 & -6 & 6 & -1 & \lfloor\underline{1} \\ 2 & -4 & 2 & 1 & \\ 2 & -2 & 0 & & \\ 2 & 0 & & & \end{array}$$

T: $g(s) = 1$



3-6. Application to Graphing. Pages 110-113.

In previous sections we learned how to find the tangent to a polynomial graph at any point on the graph. Points of particular interest are those which we might think of as peaks and valleys of the curve, that is, relative maximum and relative minimum points. Since the slope of the tangent line at these points is zero, we are particularly interested in determining all values of x for which the slope of the tangent is zero. This condition is necessary but not sufficient to insure a relative maximum or minimum point. The slope of the tangent *may* be zero at a point of inflection. To illustrate, the slope of the tangent of $f: x \longrightarrow x^3$ is zero at the point $P(0, 0)$; P is a point of inflection, not a relative maximum or minimum point.

On the other hand, we may have a point of inflection where the slope of the tangent is not zero. This section suggests one method of locating such a point; in the expansion of $f(x)$ in powers of $(x - h)$, if we set the coefficient of $(x - h)^2$ equal to zero, we locate a candidate--a possible point of inflection. If the coefficient of $(x - h)^3$ is not zero, all is well, we have spotted a point of inflection; otherwise, we must look further and examine the coefficients of higher powers of $(x - h)$. In any case, an analysis of the expansion of $f(x)$ in powers of $(x - h)$ will enable us to determine, at a glance, the behavior of the function near $x = h$.

The method of Section 3-7 provides us with a simplified scheme for locating critical points of a function; the method is a tool or short cut which should not be used prematurely to replace the procedure of this section.

Answers to Exercises 3-6. Page 113.

$f: x \longrightarrow 16 - 6x^2 + x^3$

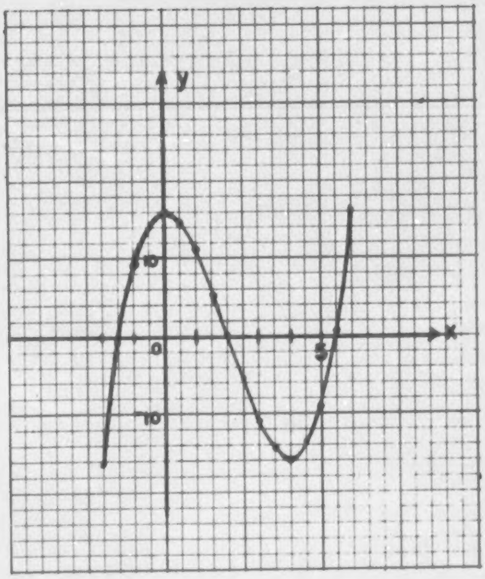
a) $3h^2 - 12h$

b) $y = 9 + 15(x + 1)$

c) If $3h^2 - 12h = 0$, $h = 0$ or $h = 4$.
(0, 16) is relative maximum point,
(4, -16) is relative minimum point,
(2, 0) is point of inflection. (Note that $\frac{0 + 4}{2} = 2$, the abscissa, and also $\frac{16 - 16}{2} = 0$, the ordinate; the point of inflection is midway between the maximum and the minimum.)

d) $f(0) = 16$, $f(2) = 0$, $f(-2) = -16$, $f(3) = -11$,
$f(-3) = -65$, $f(10) = 416$, $f(-10) = -1584$.

e)



$f: x \longrightarrow 16 - 6x^2 + x^3$

$\underline{f: x \longrightarrow 2x^3 - 4x - 1}$

a) $6h^2 - 4$

b) $y = 1 + 2(x + 1)$

c) If $6h^2 - 4 = 0$, $h = \sqrt{\frac{2}{3}}$ or $-\sqrt{\frac{2}{3}}$

$\left(\sqrt{\frac{2}{3}}, \ -\frac{8}{3}\sqrt{\frac{2}{3}} - 1\right)$ is relative minimum point,

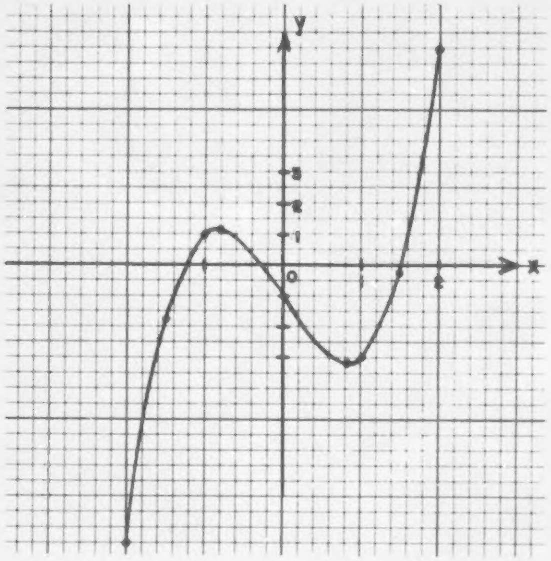
$\left(-\sqrt{\frac{2}{3}}, \ \frac{8}{3}\sqrt{\frac{2}{3}} - 1\right)$ is relative maximum point,

$(0, -1)$ is point of inflection.

d) $f(0) = -1$, $f(2) = 7$, $f(-2) = -9$, $f(3) = 41$

$f(-3) = -43$, $f(10) = 1959$, $f(-10) = -1961$.

e)



$f: x \longrightarrow 2x^3 - 4x - 1$

3-7. The Slope Function. Pages 114-116.

In this section we develop a general expression for the slope of a tangent to a polynomial graph at any point $\left(h, f(h)\right)$. In the case of a polynomial of n^{th} degree the result is

$$na_nh^{n-1} + (n-1)a_{n-1}h^{n-2} + \dots + a_1$$

where the a_i are the coefficients of the polynomial

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0.$$

At this time it would be well to review functional notation. (See Section 1-1.) "The functions $f: x \longrightarrow x^2$, $f: h \longrightarrow h^2$, $f: t \longrightarrow t^2$, ..., all describe exactly the same function, subject to our agreement that x, h, or t stand for any real number." This justifies our calling f' the slope function associated with f.

Answers to Exercises 3-7. Pages 116-117.

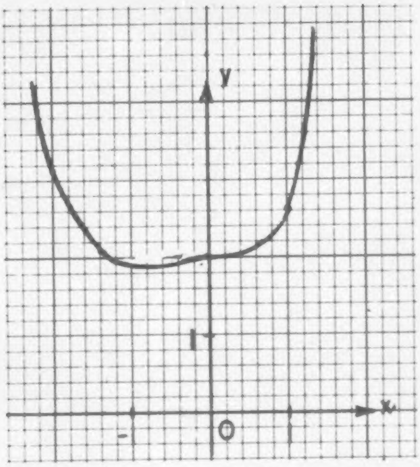
1. f: $x \longrightarrow \frac{x^4}{4} + \frac{x^3}{3} + 2$

a) $f'(x) = x^3 + x^2$

b) $f'(-1) = 0$

c) Tangent: $y = 2$

d)



f: $x \longrightarrow \frac{x^4}{4} + \frac{x^3}{3} + 2$

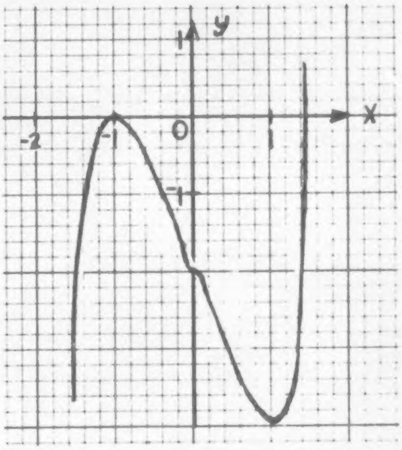
g: $x \longrightarrow 3x^5 - 5x^3 - 2$

a) $g'(x) = 15x^4 - 15x^2$

b) $g'(-1) = 0$

c) Tangent: $y = -2$

d)



g: $x \longrightarrow 3x^5 - 5x^3 - 2$

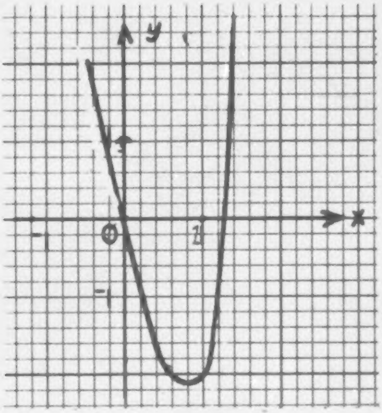
$p: x \longrightarrow x^6 - 3x$

a) $p'(x) = 6x^5 - 3$

b) $p'(-1) = -9$

c) Tangent: $y = -3x$

d)



$p: x \longrightarrow x^6 - 3x$

2. a) $f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1)$

$(-2, 13)$ = maximum point
$(1, -14)$ = minimum point

b) $f'(x) = 3x^2 - 12 = 3(x + 2)(x - 2)$
$(-2, 32)$ = maximum point
$(2, 0)$ = minimum point

c) $f'(x) = -6x^2 + 6x + 12 = -6(x - 2)(x + 1)$

$(-1, 0)$ = minimum point
$(2, 27)$ = maximum point

3. a)

$$\begin{array}{cccc|l} 2 & -6 & 6 & -1 & \underline{1} \\ 2 & -4 & 2 & 1 & \\ 2 & -2 & 0 & & \\ 2 & 0 = b_2 & & & \end{array}$$

Inflection point.

This follows since $b_2 = 0$ and $b_3 = 2 \neq 0$

b)

$$\begin{array}{cccc|l} 2 & 0 & -6 & 6 & \underline{1} \\ 2 & 2 & -4 & 2 & \\ 2 & 4 & 0 & & \\ 2 & 6 = b_2 & & & \end{array}$$

Minimum point.
(Note, that $P(1,1)$ is not on the graph.)

c) $$\begin{array}{rrrrl} 2 & -9 & 12 & -4 & \underline{1\rfloor} \\ 2 & -7 & 5 & 1 & \\ 2 & -5 & 0 & & \\ 2 & -3 & = b_2 & & \end{array}$$ Maximum point.

d) $$\begin{array}{rrrrl} 2 & -3 & -12 & 14 & \underline{1\rfloor} \\ 2 & -1 & -13 & 1 & \\ 2 & 1 & -12 & & \\ 2 & 3 & & & \end{array}$$ None of these.

4)

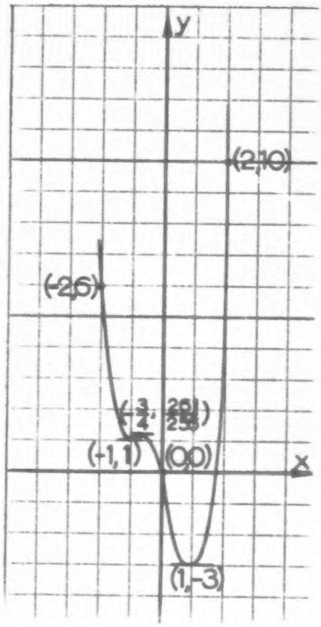
$f(x) = x^4 + x^3 - 2x^2 - 3x$

$f'(x) = 4x^3 + 3x^2 - 4x - 3 = (x^2 - 1)(4x + 3) = 0$ if and only if $x = -1, \ -\frac{3}{4}, \ +1.$

$f(-2) = 6, \quad f(0) = 0,$

$f(2) = 10, \quad f(-1) = 1,$

$f(-\frac{3}{4}) = \frac{261}{256}, \quad f(1) = -3.$



Graph of
$f: x \longrightarrow x^4 + x^3 - 2x^2 - 3x$

5. a) $f'(x) = 3x^2 - 6x$ $\quad f(1) = -1 \quad$ $f'(1) = -3$
$g'(x) = x - \frac{2}{3}$ $\quad g(1) = -1 \quad$ $g'(1) = +\frac{1}{3}$ $\}$

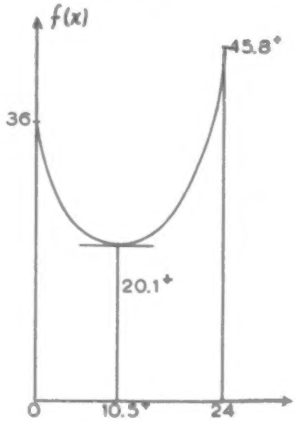
b) T_f: $y + 1 = -3(x - 1) = -3x + 3$
T_g: $y + 1 = \frac{1}{3}(x - 1)$

c) The tangent lines are perpendicular.

3-8. Maximum and Minimum Problems. Pages 117-123.

For the sake of arousing student interest, it is desirable that some of these problems be taken up. Since the most difficult part of the problem may well consist in finding the function to maximize or minimize, and since no general rule can be given for doing this, a small number of exercises is sufficient for a lesson assignment.

It is possible for a function to take on its maximum or minimum value at an end-point of the domain. In this case the slope function need not be zero. Exercise 4 is a case in point. Although the exercise does not require that the student find the maximum area, the question might be properly raised. We note that the function has a graph which resembles the figure, with a maximum value at $x = 24$ where the slope is positive, not zero. The domain is restricted by physical considerations to the interval $[0, 24]$, that is, $0 \leq x \leq 24$. The situation also arises in Example 3, where V has a minimum of the same kind at $x = 0$.



Graph of total area against circumference of circle.

Figure TC 3-8

Answers to Exercises 3-8. Pages 123-126.

1. $V = (20 - 2x)^2 x = 4(x^3 - 20x^2 + 100x) = f(x)$.
$f'(x) = 4(3x^2 - 40x + 100) = 0$, if $x = 10$ or $\frac{10}{3}$. (10 gives a minimum.) For maximum,

$$V = \frac{10}{3} \cdot \frac{40}{3} \cdot \frac{40}{3} = \frac{16,000}{27} = 592.6^+ \text{ cubic feet.}$$

2. The number to be squared will be $\frac{2}{3}N$ and the other number is $\frac{N}{3}$.

3. Fifty feet parallel to the river; 25 feet on each side.

4. If x is the length of the wire to be bent into the circle, the area is:

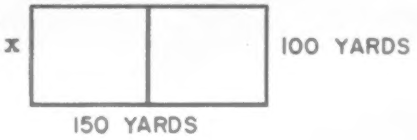
$$A = \left(\frac{24 - x}{4}\right)^2 + \frac{x^2}{4\pi} = 36 - 3x + \frac{x^2}{16} + \frac{x^2}{4\pi} = f(x)$$

$$f'(x) = -3 + \frac{x}{8} + \frac{x}{2\pi} = 0, \quad \text{if} \quad x = \frac{24\pi}{\pi + 4} \approx 10.5^+.$$

This will give a combined area for the square and circle of 20.1^+, which is a minimum.

If the maximum should be required, we should have to examine the end-points (see the general remarks on this section). The maximum area is 45.8^+ and corresponds to the case in which the entire wire is bent into the circle. (Properly speaking, the wire is not cut.)

5.



$$A = x\left(\frac{600 - 3x}{2}\right)$$

6. $G.I. = N[1.50 - (N - 10).03] = 1.80N - .03N^2$ where $N =$ number of thousands.

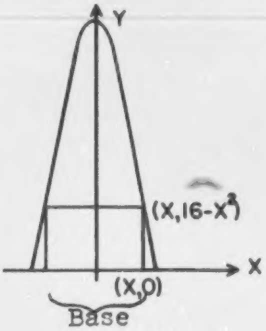
$$1.80 - .06N = 0, \quad \text{if} \quad N = 30.$$

30,000 labels will produce maximum gross income for printer.

7. The dimensions will be (height × width × length) =

$$(6 - 2\sqrt{3}) \times (4\sqrt{3}) \times (12 + 4\sqrt{3})$$

8. $(\frac{8\sqrt{3}}{3}) \times (\frac{32}{3})$. The base of the rectangle is twice the value of x.



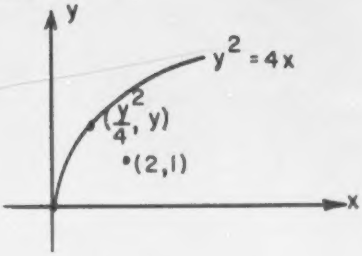
9. $D = \sqrt{(\frac{y^2}{4} - 2)^2 + (y - 1)^2}$

$K = D^2 = \frac{y^4}{16} - 2y + 5 = f(y)$

$f'(y) = \frac{y^3}{4} - 2 = 0$ if $y^3 = 8$,

$y = 2$.

So the point is $(1, 2)$.



10. If x represents the number of weeks and P the profit, then $P = (100 + 20x)(5 - \frac{x}{4}) = 500 + 75x - 5x^2 = f(x)$,

$f'(x) = 75 - 10x$

$f'(x) = 0 \Longleftrightarrow x = 7\frac{1}{2}$.

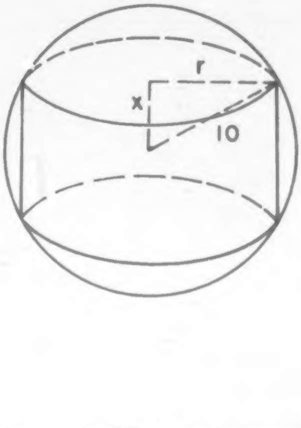
The answer is seven weeks; the eighth week will not add to his profit. (The number $7\frac{1}{2}$ is not an answer, since it is not in our domain.)

11. $V = 2\pi r^2 x = 2\pi(100 - x^2)x$

$= 200\pi x - 2\pi x^3 = f(x)$

$f'(x) = 200\pi - 6\pi x^2$

$f'(x) = 0 \Longleftrightarrow x = 10/\sqrt{3}$



Radius of inscribed right circular cylinder = $\frac{10\sqrt{6}}{3}$ inches.

Height of inscribed right circular cylinder = $\frac{20\sqrt{3}}{3}$ inches.

12. If x represents the number of additional tree plantings, then the crop

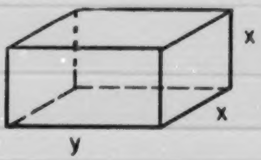
$$C = (30 + x)(400 - 10x) = 12{,}000 + 100x - 10x^2 = f(x),$$

$$f'(x) = 100 - 20x = 0 \Longleftrightarrow x = 5.$$

Total number of trees per acre is 35.

13. Girth = distance around package = $4x$.

$$y + 4x = 84$$



$$V = x^2y = x^2(84 - 4x)$$

$$= 84x^2 - 4x^3 = f(x).$$

$$f'(x) = 168x - 12x^2$$

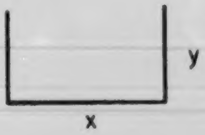
$$f'(x) = 0 \Longleftrightarrow x = 0 \quad \text{or} \quad x = 14.$$

Maximum volume of the package is 14" × 14" × 28".

14. Volume is proportional to cross-sectional area,

$$xy = x\left(\frac{10 - x}{2}\right) = 5x - \frac{x^2}{2} = f(x)$$

$$f'(x) = 5 - x = 0 \Longleftrightarrow x = 5 \quad \text{and} \quad y = 2.5$$



15. Profit = $(3x + 6)(2 - \frac{x}{3}) = -x^2 + 4x + 12 = f(x)$, where x represents the number of weeks he should wait.

$f'(x) = -2x + 4 = 0 \Longleftrightarrow x = 2$. He should ship in 2 weeks, or July 15.

16. $A = \left(\frac{p - 2x}{2}\right)x = \frac{p}{2}x - x^2 = f(x)$

$$f'(x) = \frac{p}{2} - 2x = 0 \Longleftrightarrow x = \frac{p}{4}.$$

If each side is $p/4$, the rectangle is a square.

17. Area of rectangle = $2y(4 - x) = 2y(4 - \frac{y^2}{8}) = 8y - \frac{y^3}{4} = f(y)$.

$$f'(y) = 8 - \frac{3}{4}y^2 = 0 \Longleftrightarrow y = \sqrt{\frac{32}{3}}.$$

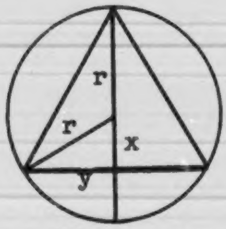
Maximum area $= \frac{64}{9}\sqrt{6},$ at $x = 4/3,$ $2y = \frac{8}{3}\sqrt{6},$ $4 - x = 8/3.$

18. $D^2 = (1 - x)^2 + (y^2) = (1 - x)^2 + \frac{8 - x^2}{4}$

$= 3 - 2x + \frac{3}{4}x^2 = f(x)$

$f'(x) = -2 + \frac{3}{2}x = 0 \iff x = \frac{4}{3},\quad y = \frac{1}{3}\sqrt{14}.$

The point $(\frac{4}{3}, \frac{1}{3}\sqrt{14})$ is nearest to $(1, 0)$.



19. $V = \frac{\pi}{3}y^2(r + x) = \frac{\pi}{3}(r^2 - x^2)(r + x)$

$= \frac{\pi}{3}(r^3 + r^2x - rx^2 - x^3) = f(x).$

$f'(x) = \frac{\pi}{3}(r^2 - 2rx - 3x^2) = 0$

$\iff x = \frac{r}{3},\quad h = r + x = \frac{4}{3}r.$

20. If we take x as the number of dollars added to the rent, the profit $= (80 - \frac{x}{2})(54 + x) = 4320 + 53x - \frac{x^2}{2} = f(x),$ and $f'(x) = 53 - x = 0 \iff x = 53.$
Since $x/2$ must be an integer, we use either 52 or 54 as rent increase, so that the rent is either \$112 or \$114 per month. (Again the answer obtained is not in our domain, so the nearest members of the domain have to be checked. See also Ex. 10.)

3-9. Newton's Method. Pages 126-130.

In Chapter 2 we developed theoretical and practical methods for finding zeros of polynomial functions. In this section we are primarily concerned with irrational zeros. It is possible to obtain an approximation to the zero(s) from inspection of the graph. Horner's method is sometimes used to estimate zeros to any desired degree of accuracy. However, in this text, Newton's method has been selected for the following reasons:

1. It is an application of the slope function.

2. Horner's method applies only to polynomial equations; Newton's applies not only to more general types of algebraic equations but also to transcendental equations.
3. It usually converges to the zero very rapidly.
4. Unlike Horner's method, it is self-correcting, in that an error made at any stage of the computation will automatically be corrected in subsequent stages.

In using Newton's method a good first guess (or approximation for the zero) decreases the number of applications which are necessary. We intuitively suspect that the method converges rapidly for large values of $f'(x_1)$, small values of $f(x_1)$, and small curvature. This is indeed true. To obtain a satisfactory first approximation it is often desirable to use straight line interpolation. (See T.C. Section 2-7.) Thus, if the function changes sign between two successive integers n_1 and n_2, we approximate the zero of the function by x_1, the abscissa of the point where the straight line through $A\big(n_1, f(n_1)\big)$ and $B\big(n_2, f(n_2)\big)$ crosses the x-axis. (See Figure TC 3-9a.)

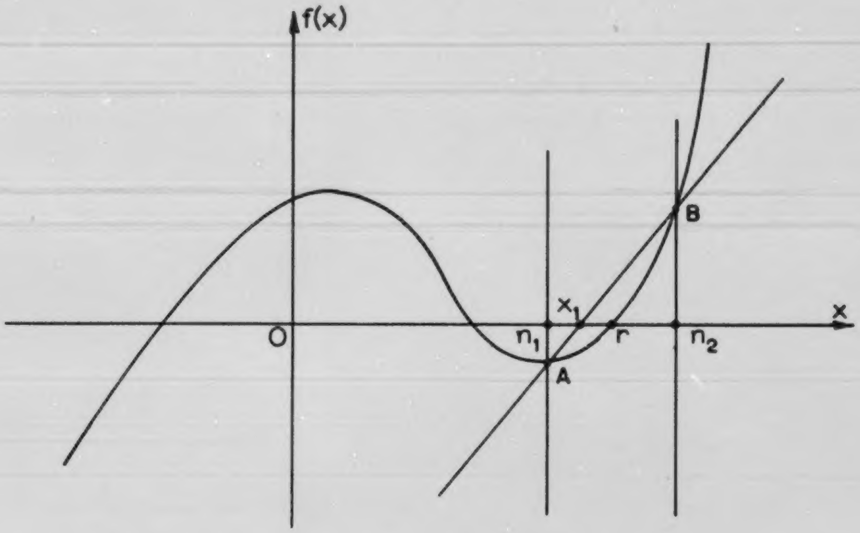


Figure TC 3-9a

Using straight line interpolation to locate the zero r.

It is readily seen that this approximation is too small if the graph G is concave upward throughout the interval $[n_1, n_2]$, and too large if the curve is concave downward throughout this interval.

To illustrate the procedure we use straight line interpolation to approximate that zero of the transcendental function

$$f: x \longrightarrow 2^x - 2x^3 + 3x^2 + 12x - 1$$

which is in the interval [3, 4]. (Note. This exercise is taken from Chapter 4, Miscellaneous Exercises.)
Since $f(3) = 16$ and $f(4) = -17$, the function f has a zero which is between 3 and 4. (See Figure TC 3-9b.)

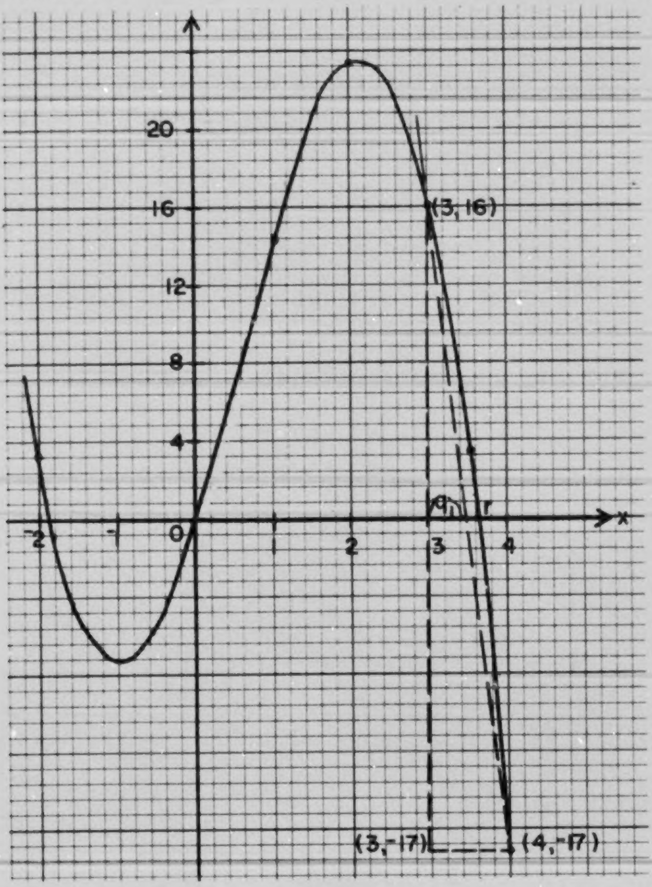


Figure TC 3-9b
A portion of the graph of $x \longrightarrow 2^x - 2x^3 + 3x^2 + 12x - 1$

By similar triangles $\frac{q_1}{1} = \frac{16}{33} \approx 0.5$, and our first approximation is $3 + 0.5 = 3.5$. We may use straight line interpolation again (See Figure TC 3-9c) to obtain q_2. Thus $q_2/0.1 \approx 3.3/3.4 \approx 0.97$ or $q_2 \approx 0.10$ and our second straight line approximation is $3.5 + 0.1 = 3.6$, to the nearest one-tenth.

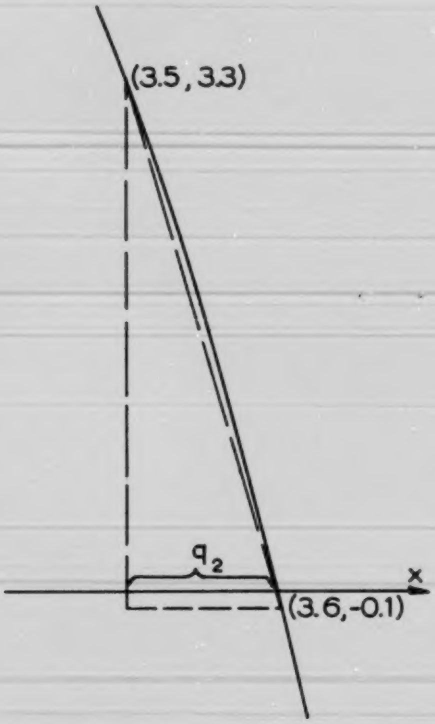


Figure TC 3-9c
A portion of the graph of $x \longrightarrow 2^x - 2x^3 + 3x^2 + 12x - 1$

Answers to Exercises 3-9. Page 131.

1. $x^2 - 2 = 0 \qquad x = 1.414$

2. $f(x) = x^3 - 12x + 1 = 0 \qquad x_1 = 0$

$f(0) = 1$

$f'(x) = 3x^2 - 12 \qquad f'(0) = -12$

$$x_2 = 0 - \frac{1}{-12} \approx 0.08$$

$$\begin{array}{rrrrl} 1 & 0 & -12 & 1 & \underline{|0.08} \\ & .08 & .0064 & -.959488 & \\ \hline 1 & .08 & -11.9936 & .040512 = f(.08) & \end{array}$$

$$\begin{array}{rrrl} 3 & 0 & -12 & \underline{|.08} \\ & .24 & .0192 & \\ \hline 3 & .24 & -11.9808 = f'(.08) & \end{array}$$

$$x_3 = 0.08 - \frac{0.040512}{-11.9808} \approx 0.08 + 0.0034 \approx 0.083$$

3. $f(x) = x^3 - 3x^2 + 2 \qquad f'(x) = 3x^2 - 6x$

$$\left.\begin{array}{l} f(2) = -2 \\ f(3) = 2 \end{array}\right\} \text{try } x_1 = 2.7$$

$$\begin{array}{rrrrl} 1 & -3 & 0 & 2 & \underline{|2.7} \\ & 2.7 & -.81 & -2.187 & \\ \hline 1 & -0.3 & -.81 & -0.187 = f(2.7) & \end{array}$$

$$\begin{array}{rrrl} 3 & -6 & 0 & \underline{|2.7} \\ & 8.1 & 5.67 & \\ \hline 3 & 2.1 & 5.67 = f'(2.7) & \end{array}$$

$$x_2 = 2.7 - \frac{-0.187}{5.67} \approx 2.7 + .033 \approx 2.73$$

4. $f(x) = x^3 + 3x - 7 \qquad f'(x) = 3x^2 + 3$

$$\left.\begin{array}{l} f(1) = -3 \\ f(2) = 7 \end{array}\right\} \text{ take } x_1 = 1.4$$

$$\begin{array}{cccc|l} 1 & 0 & 3 & -7 & \underline{1.4} \quad f'(1.4) = 3(2.96) = 8.88 \\ & 1.4 & 1.96 & 6.944 & \\ \hline 1 & 1.4 & 4.96 & -0.056 = f(1.4) & \end{array}$$

$$x_2 = 1.4 - \frac{-0.056}{8.88} \approx 1.406(3)$$

$$\begin{array}{cccc|l} 1 & 0 & 3 & -7 & \underline{1.41} \quad f'(1.41) = 8.9643 \\ & 1.41 & 1.9881 & 7.033221 & \\ \hline 1 & 1.41 & 4.9881 & .033221 = f(1.41) & \end{array}$$

$$x_3 = 1.41 - \frac{0.033221}{8.9643} \approx 1.406$$

3-10. The Graph of Polynomial Functions Near Zeros of Multiplicity Greater Than One. Pages 131-135.

This section is important in the sense that it provides an opportunity for a review and extension of Chapter 2 by integrating ideas and concepts of that chapter with the notion of the slope function.

By way of review we recall that if two polynomial functions f and g have the same zeros, it does not follow that the graph of f is the same as the graph of g; indeed $f(x)$ is not necessarily equal to $g(x)$ for all x. For example,

$$f: x \longrightarrow x^3 - 3x - 2$$
$$g: x \longrightarrow x^4 - 2x^3 - 3x^2 + 4x + 4$$

where f has degree 3 while g has degree 4, although f has two zeros, -1 and 2, and g has the same two zeros, -1 and 2.

Furthermore, we note that each of the equations

$$f(x) = x^3 - 3x - 2 = (x + 1)^2(x - 2) = 0$$

and $$g(x) = x^4 - 2x^3 - 3x^2 + 4x + 4 = (x + 1)^2(x - 2)^2 = 0$$

has the solution set $\{-1, 2\}$.

A study of the shape of a graph leads quite naturally to consideration of the concavity of a curve. In Chapter 3 we make no formal mention of the direction of concavity of a graph; however we have included graphs of functions and their associated slope functions to encourage an intuitive approach to the notion. (See Figures 3-10a and 3-10c.)

To facilitate the following discussion, we have reproduced Figure 3-10a from the Text (page 132).

Example 1. $f: x \longrightarrow (x + 1)^2(x - 2)$. Over what interval is the slope of the tangent to the graph of f negative?

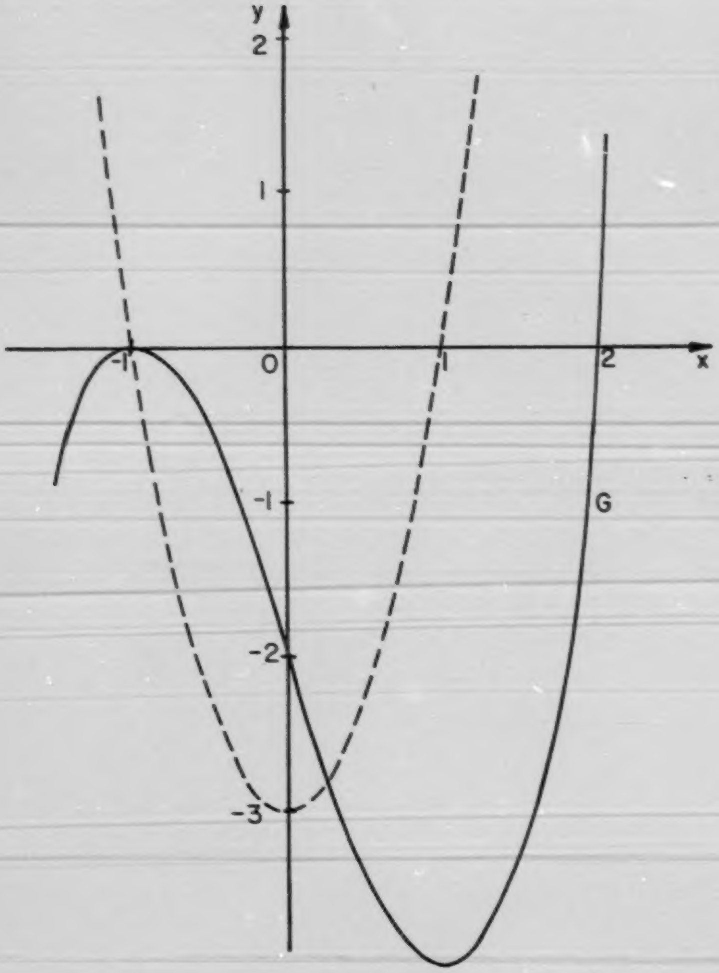
Solution. The graph of f'(Figure 3-10a) clearly shows that $f'(x) < 0$ for $-1 < x < 1$. Thus the slope of the graph G is negative for $-1 < x < 1$.

What other information does the graph of the slope function f' give?

a) We note that f' is steadily decreasing for $x < 0$ and reaches a minimum value $f'(0) = -3$ at $x = 0$. In terms of graph G this means that the slope of G decreases steadily throughout the interval $x < 0$. At the point where G crosses the y-axis its slope has a minimum value, -3.

b) For all $x > 0$, the graph of f' increases steadily. Thus, for $x > 0$, the slope of G steadily increases. In other words, at the point $(0, -2)$ the direction of concavity of G changes; to the left of $(0, -2)$ G is (said to be) concave downward, to the right of $(0, -2)$ G is concave upward, hence the point $(0, -2)$ is a point of inflection of the graph G.

Generalizing, we observe that points of inflection of polynomial graphs may be located by finding values of x for which the slope function f' has a relative maximum or minimum value.



G is the graph of f: $x \longrightarrow (x + 1)^2(x - 2)$
Graph of f': $x \longrightarrow 3(x + 1)(x - 1)$ is indicated by the dotted line.

Figure T.C. 3-10a

There is another important observation which we shall make. (See Figures 3-6, 3-10a, and 3-10c.) If the graph of a polynomial function meets the x-axis at points A and B, then at least one of the tangents to the arc AB of the graph must be parallel to the x-axis. A proof of this statement is not given since it is based on the continuity of the function and the existence of the slope function throughout the interval AB. Intuitively, however, the statement is quite reasonable. For if G does not coincide with the x-axis, then it is above (or below) the x-axis for at least one x in this interval and reaches a maximum (or minimum) value at some point in the interval. At such a point the tangent must be horizontal. (See Figure T.C. 3-10b.)

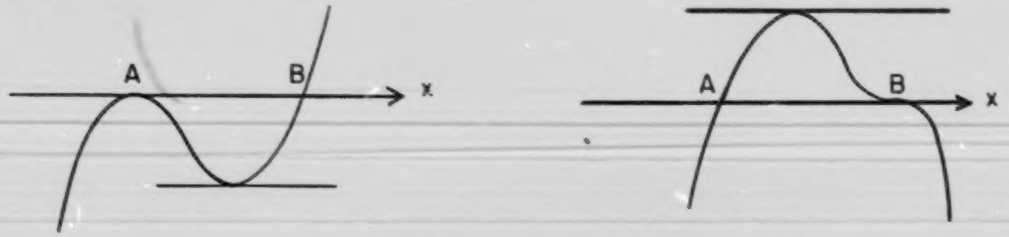


Figure T.C. 3-10b

The theorem may be restated thus: If a and b are zeros of a polynomial function f, then the related slope function f' has at least one zero in the interval between $x = a$ and $x = b$. This means that the graph of the slope function f' has a point in common with the x-axis between points A and B.

For example, the function $p: x \longrightarrow x^3 + x^2 - 24x + 36$ has zeros -6, 2 and 3; the associated slope function p' has zeros in the intervals [-6, 2] and [2, 3]. These zeros are $\frac{-1 \pm \sqrt{73}}{3}$ or -3.2 and 2.5, approximately.

We observe, however, that -1 is a zero of $f: x \longrightarrow (x + 1)^2(x - 2)$ and -1 is also a zero of the slope function $f': x \longrightarrow 3(x + 1)(x - 1)$. In the case of f, -1 is a zero of multiplicity two; in the case of f', -1 is a zero of multiplicity one.

Consider also $g: x \longrightarrow (x - 1)^3(x^2 + 1)$ and the related slope function $g': x \longrightarrow (x - 1)^2(5x^2 - 2x + 3)$. In this case note that +1 is a zero of multiplicity three of function g and a zero of multiplicity two of the related slope function g'.

We may generalize this observation and state, without proof, that any zero r of multiplicity n greater than one of the polynomial function f is also a zero of multiplicity $n - 1$ of the associated function f'. This result may be used to assist in locating zeros of polynomial functions.

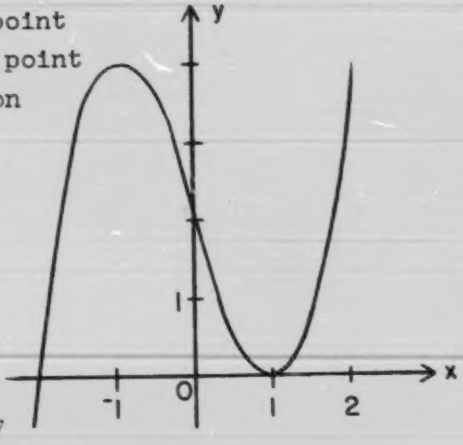
Example 2.

To find the zeros of $f: x \longrightarrow 12x^3 + 28x^2 + 3x - 18$, given that it has a zero of multiplicity greater than one, we examine the related slope function $f': x \longrightarrow 36x^2 + 56x + 3$. Since $f'(x) = (2x + 3)(18x + 1)$, the zeros of f' are $-3/2$ and $-1/18$. But f has a zero r of multiplicity greater than one, hence one of the zeros of f' must also be the required zero r of f. In other words either $(2x + 3)^2$ or $(18x + 1)^2$ is a factor of $f(x) = 12x^3 + 28x^2 + 3x - 18$. We readily see that $f(x) = (2x + 3)^2(3x - 2)$. Hence the zeros of f are $-3/2$ and $2/3$. Graphs of the function f and the related slope function f' show clearly that the graph of f' crosses the x-axis at the point where f has the zero $-3/2$ of multiplicity two. The point $(-3/2, 0)$ is a relative maximum point of the graph of f. This may be verified by the expansion of $f(x)$ in powers of $(x + 3/2)$: $f(x) = -26(x + 3/2)^2 + 12(x + 3/2)^3$.

Answers to Exercises 3-10. Page 136.

1. $$f(x) = x^3 - 3x + 2$$
$$= (x - 1)^2(x + 2)$$

   a) 2, 1 (of multiplicity two)
   b) (1, 0) is relative minimum point
      (-1, 4) is relative maximum point
      (0, 2) is point of inflection
   c) graph: $x \longrightarrow x^3 - 3x + 2$



$x \longrightarrow x^3 - 3x + 2$

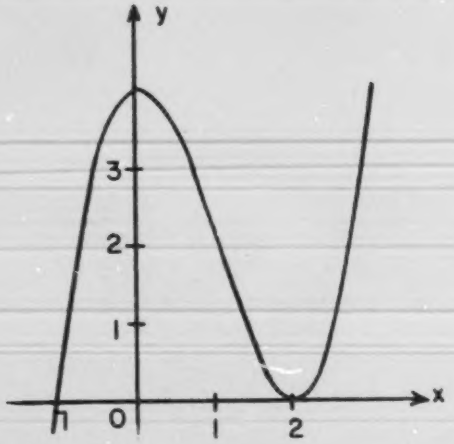
2. Zeros are 2(of multiplicity two) and -1

$f'(0) = 0$ and $f'(2) = 0$

(0, 4) is relative maximum point

(2, 0) is relative minimum point

graph: $x \longrightarrow x^3 - 3x^2 + 4$



$x \longrightarrow x^3 - 3x^2 + 4$

3. a) $f(x) = (x + 1)^2(x^2 + 1)^2 = x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1$

P(-1, 0) is a relative minimum point.

b) $f(x) = x^6 - 2x^5 + 3x^4 - 4x^3 + 3x^2 - 2x + 1 = (x-1)^2(x^2+1)^2$

P(1, 0) is a relative minimum point.

c) $f(x) = (x - 1)^3(x^2 + 1)$

P(1, 0) is a point of inflection.

d) $f(x) = (x - 1)^3(x + 1)(x^2 + 1)$

P(1, 0) is a point of inflection.

Solutions to Miscellaneous Exercises of Chapter 3. Pages 138-144.

1. $f(x) = 2x^2 - 7x - 4$
   $f'(x) = 4x - 7$
   $x_2 = x_1 + 2$, hence $f'(x_2) = 4(x_1 + 2) - 7 = 4x_1 + 1$.
   $f'(x_2) - f'(x_1) = (4x_1 + 1) - (4x_1 - 7) = 8$

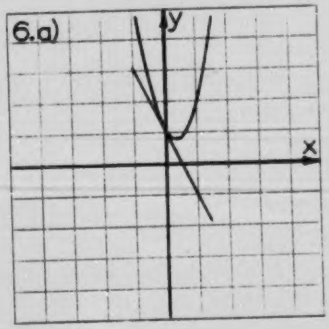
2. $f(x) = ax^2 + bx + 8$
   $f'(x) = 2ax + b$
   $f'(-3) = -6a + b = -1$
   $f(-3) = 9a - 3b + 8 = 2$
   $\therefore\ a = 1,\quad b = 5$

3. $f(x) = ax^2 + bx + c$
   $f(6) = 36a + 6b + c = 0$
   $f(2) = 4a + 2b + c = 0$
   $f'(x) = 2ax + b$
   $f'(6) = 12a + b = 5$
   $\therefore\ a = 5/4,\quad b = -10$
   $f'(2) = -5$

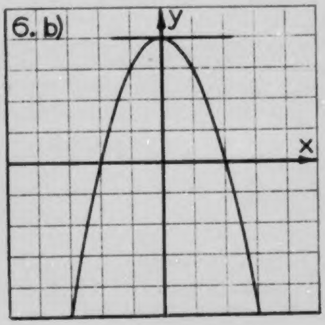
4. $f(x) = x^2 - 4x + 3 = f(h) + (2h - 4)(x - h) + (x - h)^2$
   Hence linear approximation is $y = f(h) + (2h - 4)(x - h)$.
   This becomes $y = -1$ if $h = 2$, and $y = 3 + 4(x - 4)$ if $h = 4$.
   Error involved is $f(x) - y = (x - h)^2$.
   At $x = 2.01$ error is $(2.01 - 2)^2 = 0.0001$.
   At $x = 4.01$ error is $(4.01 - 4)^2 = 0.0001$.
   If $(x - h)^2 < 0.01$, $|x - h| < 0.1$.

5. $f(x) = 5x^2 + (1 - 10h)x + (3 - h + 5h^2)$

6. a) Tangent at $(0, 1)$ has equation $y = 1 - 2x$.



b) Tangent at $(0, 4)$ has equation $y = 4$.



7. $f(x) = -35 + 45(x + 2) - 17(x + 2)^2 + 2(x + 2)^3$

8. $f(x) = a(x - h)^2 + b(x - h) + c = ax^2 + (-2ah + b)x + (ah^2 - bh + c)$

$\therefore\ f(1) = a + (-2ah + b) + ah^2 - bh + c = ah^2 + (-2a - b)h + (a + b + c)$

Expanding $f(x)$ in powers of $(x - 1)$ we have

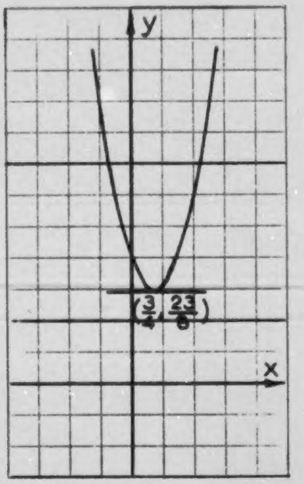
$$f(x) = a(x-1)^2 + (2a+b-2ah)(x-1) + ah^2 + (-2a-b)h + (a+b+c)$$

$$= a(x - 1)^2 + (2a + b - 2ah)(x - 1) + f(1).$$

9. $f(x) = 4 - 3x + 2x^2$

Tangent at $\big(h, f(h)\big)$: $y = f(h) + (4h - 3)(x - h)$.

Tangent is horizontal if $4h - 3 = 0$, that is at $h = 3/4$.



10. $f: x \longrightarrow a_2x^2 + a_1x + a_0 = b_2(x - h)^2 + b_1(x - h) \div b_0$

where $b_2 = a_2$.

The necessary condition for a point of inflection is that $b_2 = 0$.

But $b_2 = a_2 \neq 0$.

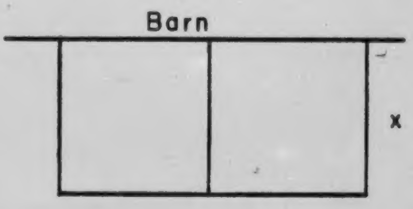
11. Let $f(x) = 2x^3 - 5x^2 + 3x - 4$, then expressing $f(x)$ in powers of $(x - 3)$ we have
$f(x) = 14 + 27(x - 3) + 13(x - 3)^2 + 2(x - 3)^3$

12. a) Area = $x(120 - 3x)$

$f(x) = 120x - 3x^2$

$f'(x) = 120 - 6x$.

If $f'(x) = 0$, $x = 20$,
$120 - 3x = 60$.
Area is 1200 square feet.

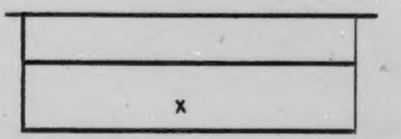


b) Area = $x(\frac{120 - 2x}{2})$

$= 60x - x^2 = f(x)$

$f'(x) = 60 - 2x$.
If $f'(x) = 0$, $x = 30$, $\frac{120 - 2x}{2} = 30$.

Area is 900 square feet.



13. $d = 88t - 16t^2$. If $d = 0$, $t = 0$ or $11/2$.

$f(t) = 88t - 16t^2$

$f'(t) = 88 - 32t$. If $f'(t) = 0$, $t = 11/4$.

Answer: In 11/4 seconds, it will reach a height of 121 feet. It will hit the ground in 11/2 seconds.

14. $g: x \longrightarrow x^2 - 3x + 4$

$g'(x) = 2x - 3$, $g'(0) = -3$

$f'(x) = 6x^2 + 2x - 3$, $f'(0) = -3$

Thus $g'(0) = f'(0)$.

15. If $f(x) = x^7 + a^7$, then $f(-a) = a^7 - a^7 = 0$, and $(x + a)$ is a factor.

16. $ax_1^{\ 2} + bx_1 + c = 0$

$ax_2^{\ 2} + bx_2 + c = 0$

Thus, $a(x_1^{\ 2} - x_2^{\ 2}) + b(x_1 - x_2) = 0$

or $a(x_1 + x_2) + b = 0$ for $x_1 \neq x_2$.

But $f'(x) = 2ax + b = 0$

for $x = \frac{-b}{2a} = \frac{a(x_1 + x_2)}{2a} = \frac{x_1 + x_2}{2}$.

Hence f has a minimum at $x = \frac{x_1 + x_2}{2}$.

17. $x_1 = x_2 = x_3$, etc.

18. $x = 0.35$ to two decimals.

19. $g(x) = ax^2 + bx + c$, $g(2) = 4a + 2b + c = 3$,

$g'(x) = 2ax + b = 3x + 2$.

Hence $a = 3/2$, $b = 2$, $c = -7$ and $g(x) = \frac{3}{2}x^2 + 2x - 7$.

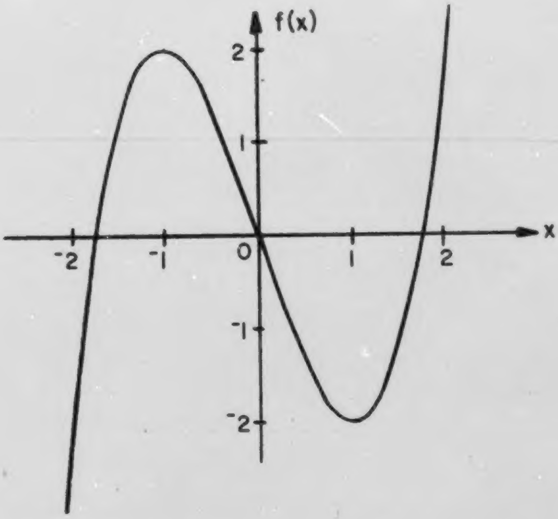
20. $f(x) = x^2 + 5x + 4$

$f'(x) = 2x + 5 = -5 \Longleftrightarrow x = -5$; $f(-5) = 4$.

21. $f(x) = ax^2 + bx + c = ax^2 + 4x - 3, \quad a \neq 0.$

22. $f(x) = -3x + x^3$

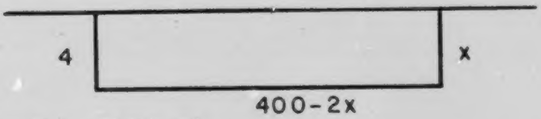
At $(0, 0)$ tangent has equation $y = -3x$. Graph is below tangent to left of $(0, 0)$ and above tangent at right of $(0, 0)$ for $|x|$ small. Hence $(0, 0)$ is point of inflection.



23. $A = x(400 - 2x) = f(x)$

$f'(x) = 400 - 4x$

If $x = 100$, pasture is 200 yd. × 100 yd.

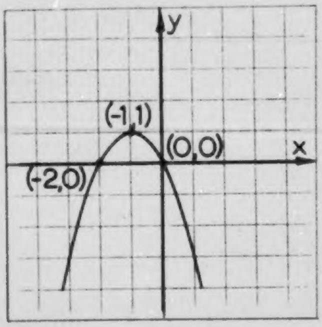


24. Volume $= C = yx(mx + b)$,

hence $y = \frac{C}{x(mx + b)} = f(x)$.

25. a) $f: x \longrightarrow ax^2 + bx + c$

$f': x \longrightarrow 2ax + b$



$f(0) = c = 0,$

$f'(-1) = -2a + b = 0,$

$f(-1) = a - b = 1,$

hence $a = -1,\ b = -2$

$f(x) = -x^2 - 2x.$

Alternative Solution:

$f(x) = 1 + 0(x+1) + a(x+1)^2,$

$a < 0$

$f(0) = 1 + a = 0 \Longleftrightarrow a = -1$

b) $g: x \longrightarrow x^2 + 2x + 2$ is symmetric to f with respect to the line $y = 1$; the graph of g has a minimum at $(-1, 1)$ and passes through the point $(0, 2)$.

26. $f(x) = ax^2 + bx + c$

$f(0) = c = 0$

If $f'(x) = 2ax + b = 0$, $x = \frac{-b}{2a}$, and $b = -4a$.

Since $f(2) = 4a - 8a = 3$, $a = -3/4$, $b = 3$;

thus $f(x) = -\frac{3}{4}x^2 + 3x.$

Alternatively,

$f(x) = 3 + a(x - 2)^2$

$f(0) = 3 + 4a = 0$

$a = -3/4$

$f(x) = 3 - \frac{3}{4}(x - 2)^2$

27. $f(x) = 4x^2 + 3x - 2$

$y = 3x - 2$ is tangent line at point $(0, -2)$.

If $4x^2 + 3x - 2$ is replaced by $3x - 2$, the error $= 4x^2$. If error is to be less than 0.01, $4x^2 < 0.01$, or $|x| < 0.05$.

Answer: $-0.05 < x < 0.05$.

28. $f: x \longrightarrow ax^3 + bx^2 + cx + d$

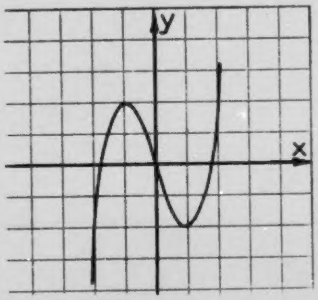
$f(1) = a + b + c + d = -2$

$\underline{f(-1) = -a + b - c + d = 2}$

$2b \qquad + 2d = 0 \Longrightarrow b = -d$

$2a \ + \ 2c \qquad = -4 \Longrightarrow a = -c - 2$ or $c = -a - 2.$

But $f(0) = 0.$ So $d = 0$ and $b = 0.$



$f: x \longrightarrow ax^3 - x(a + 2),$

$f': x \longrightarrow 3ax^2 - (a + 2)$, and $f'(1) = 3a - (a + 2) = 0.$

$2a = 2, \quad a = 1,$ and $c = -3.$

So, $f: x \longrightarrow x^3 - 3x,$

$\quad f': x \longrightarrow 3x^2 - 3.$

$f'(0) = -3$ and $(0,0)$ is a point of inflection.

$\left.\begin{array}{l} f(h) = 3h^2 - 3 \\ f(-h) = 3h^2 - 3 \end{array}\right\}$ $\therefore$ slopes at $x_1 = h$ and $x_2 = -h$ are the same.

29. $g(x) = -3(x - 3) + (x - 3)^3$

$g'(3) = -3$; $(3, 0)$ is a point of inflection.

$g'(x) = 3x^2 - 18x + 24 = 3(x - 2)(x - 4)$

$g'(3 + k) = 3(k + 1)(k - 1), \quad g'(3 - k) = 3(1 - k)(-1 - k),$

hence, $\quad g'(3 + k) = g'(3 - k) = 3(k^2 - 1).$

(See figure on next page.)

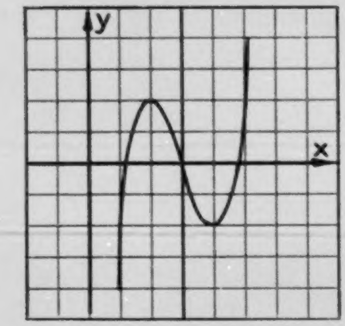


Figure to Exercise 29.

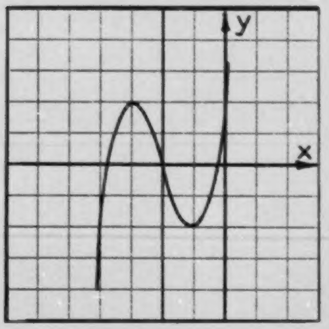


Figure to Exercise 30.

30. $$\left.\begin{array}{l} f:\ x \longrightarrow x^3 - 3x \\ g:\ x \longrightarrow (x - 3)^3 - 3(x - 3) \end{array}\right\} \Longrightarrow g(x) = f(x - 3) \Longrightarrow h = -3$$

$$g_1:\ x \longrightarrow (x + 2)^3 - 3(x + 2) = x^3 + 6x^2 + 9x + 2$$

31. f is symmetric with respect to $(0, 0)$, the point of inflection, since for every point $\big(x, f(x)\big)$ on the graph there is a corresponding point $\big(-x, -f(x)\big)$; that is $f(-x) = -f(x)$.
Similarly, g is symmetric with respect to $(3, 0)$, the point of inflection, since for every point $\big(3 + x,\ g(3 + x)\big)$ on the graph of g there is a corresponding point $\big(3 - x, -g(3 + x)\big)$; in other words $g(3 - x) = -g(3 + x)$.

32. $f(x) = x^3 - 6x^2 + 9x - 4$

$f(x) = -3(x - 1)^2 + (x - 1)^3$ $(1, 0)$ relative maximum

$f(x) = -2 - 3(x - 2) + (x - 2)^3$ $(2, -2)$ point of inflection

$f(x) = 3(x - 3)^2 + (x - 3)^3$ $(3, -4)$ relative minimum.

33. A(7), B(8), C(5), D(6).

34. a) $f(x) = 0 + 0(x - 2) + (x - 2)^2$
(2, 0) is minimum point.

b) $f(x) = 0 + 0(x - 2) + 0(x - 2)^2 - (x - 2)^3$

(2, 0) is point of inflection with horizontal tangent.

c) $f(x) = 0 + 0(x - 2) + 0(x - 2)^2 + 0(x - 2)^3 + (x - 2)^4$

(2, 0) is minimum point.

d) $f(x) = (x - 1)^2[(x - 1) + 3]$

$= 0 + 0(x - 1) + 3(x - 1)^2 + (x - 1)^3$

(1, 0) is a relative minimum point.

Also, $f(x) = [(x + 1) - 2]^2[(x + 1) + 1]$

$= 4 + 0(x + 1) - 3(x + 1)^2 + (x + 1)^3$

(-1, 4) is a relative maximum point.

35. $x \approx 0.618$.

36. $f(x) = x^3 - 3x^2 + 2x - 1$

$f'(x) = 3x^2 - 6x + 2 = 0$ when $x = \frac{6 \pm \sqrt{12}}{6} = \frac{3 \pm \sqrt{3}}{3}$.

For $\frac{3 - \sqrt{3}}{3}$ we get a relative maximum which is below zero, and this means that there is only one real root, $x \approx 2.325$.

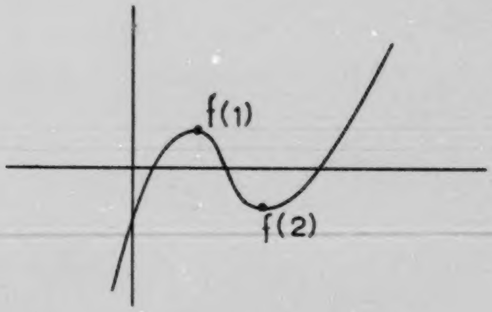
37. a) No value of k since cubic equations will always have at least one real root. (Imaginary roots must occur in pairs.)

b) For $k > -4$ or $k < -5$.

c) For $-5 < k < -4$.

Solution.

First find $f'(x) = 6x^2 - 18x + 12 = 0$ if $x = 1$ or $x = 2$. $f(1) = 5 + k$ and $f(2) = 4 + k$. $\big(1, f(1)\big)$ is a relative maximum. $\big(2, f(2)\big)$ is a relative minimum. (Continued on following page.)

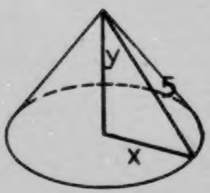


For one real root $f(1) < 0$ or $f(2) > 0$, that is, $5 + k < 0$ or $4 + k > 0$.

For three real roots $f(1) > 0$ and $f(2) < 0$, that is, $k + 5 > 0$ and $k + 4 < 0$.

38. $f(0) = f(k) = ah^2 - bh + c$

39.



$$x^2 + y^2 = 25 \Longrightarrow x^2 = 25 - y^2$$

$$\text{Volume} = \pi x^2 \cdot \frac{y}{3} = (25 - y^2)\frac{\pi y}{3}$$

$$= \frac{25\pi y}{3} - \frac{\pi y^3}{3} = f(y).$$

$f'(y) = \frac{25\pi}{3} - \pi y^2 = 0$ if $y = \frac{5\sqrt{3}}{3}$ and $x^2 = 50/3$. So, maximum volume is $\frac{250\pi\sqrt{3}}{27}$.

40. $x_3 = 2.095$ is required approximation.

41. $f(x) = 2k + 3x - 5x^2$

$f'(x) = 3 - 10x = 0 \Longrightarrow x = 0.3$

Since $f(.3) = 2k + 3(.3) - 5(.3)^2 = 0.3$,

$k = -0.075$.

42. Difference = $x - x^2$, thus
$f(x) = x - x^2$,
$f'(x) = 1 - 2x$.
If $f'(x) = 0$, then $x = 1/2$.
Expanding $f(x)$ in powers of $(x - 1/2)$, we have

$f(x) = 1/4 - (x - 1/2)^2$
which clearly shows that we have a maximum at $x = 1/2$.

43. The point $(3, 0)$ may be determined by inspection of the graph of the circle, and the point $(5, 0)$.

Exercises 43 - 45 are included to show that not all maximum, minimum problems are solved via the slope function.

For, in the case at hand, if the distance from the point $(5, 0)$ to the circle $x^2 + y^2 = 9$ is given by D, then
$D^2 = (5 - x)^2 + (0 - y)^2 = 25 - 10x + x^2 + y^2 = 34 - 10x$.

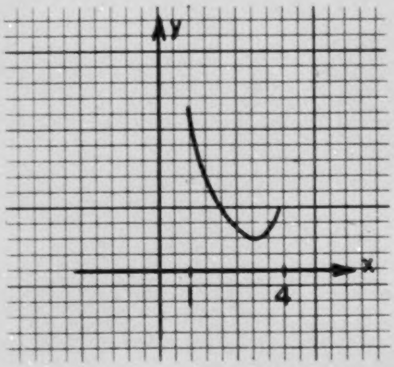
But $f: x \longrightarrow 34 - 10x$ has no critical value. On the interval $[-3, 3]$, however, D^2 has the minimum value 4 at $x = 3$. This agrees with our original result.

44. $f: x \longrightarrow \dfrac{2}{x^2 - 6x + 10}$ has a maximum value when
$g: x \longrightarrow x^2 - 6x + 10$ has a minimum value. But

$g'(x) = 2x - 6 = 0$ if and only if $x = 3$, hence the maximum value of f is $f(3) = 2$.

45. $f(x) = x^2 - 6x + 10$

$f(1) = 5$, $f(4) = 2$.
On the interval $1 \leq x \leq 4$
the maximum value is 5 at
the end point $(1, 5)$.



46. $g(x) = 6x^5 - 4x^3 + 5x + 2$

$g(x) \approx -4x^3 + 5x + 2$, $|x|$ near zero.
$g(.1) \approx -4(.001) + 5(.1) + 2 = 2.496$.

$f(x) = x^6 - x^4 + 2.5x^2 + 2x - 6$,

$f'(x) = 6x^5 - 4x^3 - 5x + 2$.

$f'(.1) \approx 2.496$, since $6(10)^{-5} = .00006 < \frac{1}{1000}$.

47. $f(x) = x^3 + 3x^2 - 4x - 3$

$f'(x) = 3x^2 + 6x - 4$

If $f'(x) = 0$, $x_1 = \frac{-6 + \sqrt{84}}{6}$, $x_2 = \frac{-6 - \sqrt{84}}{6}$.

Thus, there is a point of inflection at $\frac{x_1 + x_2}{2} = -1$.

$f'(-1) = -7$ and $f(-1) = 3$.

Tangent at $(-1, 3)$ is given by $y = 3 - 7(x + 1) = -4 - 7x$.

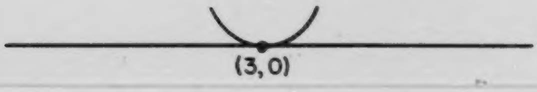
48. $g(x) = 3x^4 - 12x^3 + 12x^2 - 4$
$g'(x) = 12x(x - 1)(x - 2)$

Point $(0, -4)$ is a minimum point.
Point $(1, -1)$ is the relative maximum point.
Point $(2, -4)$ is a minimum point.

49. $f: x \longrightarrow (x - 3)^2(x + 4)^3$

$$\begin{aligned} f(x) &= (x - 3)^2[(x - 3) + 7]^3 \\ &= (x - 3)^2[(x-3)^3 + 21(x-3)^2 + 147(x-3) + 343] \\ &= (x-3)^5 + 21(x-3)^4 + 147(x-3)^3 + 343(x-3)^2 \end{aligned}$$

The point $(3, 0)$ is relative minimum point.

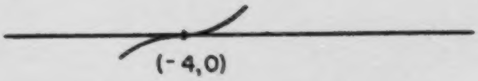


Also, $f(x) = (x + 4)^3[(x + 4) - 7]^2$

$$= (x + 4)^3[(x + 4)^2 - 14(x + 4) + 49]$$

$$= (x + 4)^5 - 14(x + 4)^4 + 49(x + 4)^3$$

Point $(-4, 0)$ is a point of inflection; tangent at $(-4, 0)$ is x-axis.



50. Root is 0.200 to three decimals. $\left(f(0.2) = 0.00032\right)$

51. $f'(x) = 3a_3x^2 + 2a_2x + a_1$

At relative maximum or minimum point $f'(x) = 0$.

$f'(x) = 0$ has roots x_1 and x_2 where $\dfrac{x_1 + x_2}{2} = \dfrac{-a_2}{3a_3}$.

Expanding $f(x)$ in powers of $(x-h)$, we have

$$f(x) = f(h) + f'(h)(x-h) + (3a_3h + a_2)(x-h)^2 + a_3(x-h)^3.$$

If $h = \dfrac{x_1 + x_2}{2} = \dfrac{-a_2}{3a_3}$, the coefficient of $(x-h)^2$

$$3a_3h + a_2 = 3a_3\left(\frac{-a_2}{3a_3}\right) + a_2 = -a_2 + a_2 = 0.$$

Thus, at $x = \dfrac{x_1 + x_2}{2}$ there is a point of inflection.

52. $f(x) = b_0 + b_1(x - h) + \ldots + b_m(x - h)^m$

$g(x) = cf(x) = cb_0 + cb_1(x - h) + \ldots + cb_m(x - h)^m$

$b_1 = f'(h)$

$cb_1 = g'(h)$

Hence $g'(h) = cf'(h)$.

Since this is true for every h, $g'(x) = cf'(x)$.

53. Write $f(x)$ and $g(x)$ as expansions in powers of $(x - h)$ and proceed as in Exercise 52.

54. Every polynomial in x can be expressed as a linear combination of x^k where k takes on non-negative integral values.

55. Since $f(x) = 10x + 3x^5 + 2x^6$, the tangent to the graph at $P_1(0, 0)$ is $y = 10x$. The graph lies below the tangent at the left of P_1 and above the tangent at the right of P_1, for $|x|$ sufficiently small.

Expanding $f(x)$ in powers of $(x + 1)$ we have

$$f(x) = -11 + 13(x+1) - 10(x+1)^3 + 15(x+1)^4 - 9(x+1)^5 + 2(x+1)^6.$$

Thus, $P_2(-1, -11)$ is a point of inflection. The graph lies above the tangent at the left of P_2 and below the tangent at the right of P_2.

56. $f(x) = x^4 - 2x^3 - 7x^2 + 10x + 10$

$f'(x) = 4x^3 - 6x^2 - 14x + 10$

$f'(5/2) = 0$. Expanding $f(x)$ in powers of $(x - 5/2)$, we have

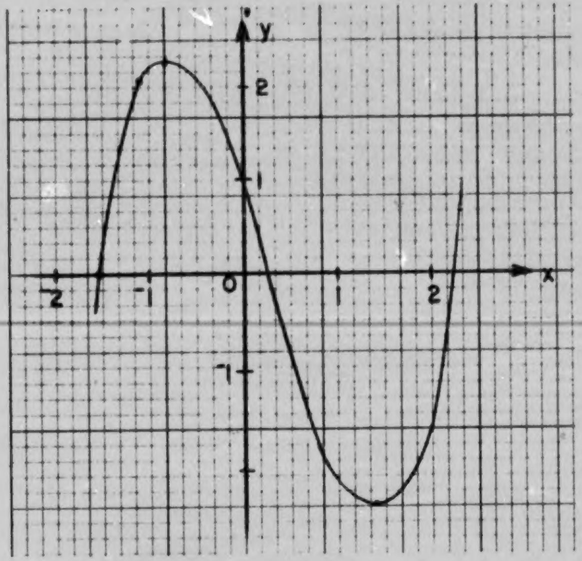
$$f(x) = -\frac{15}{16} + \frac{31}{2}(x - 5/2)^2 + 8(x - 5/2)^3 + (x - 5/2)^4,$$

showing that $P(5/2, -15/16)$ is a relative minimum point. Since $f(2) = 2$, $f(5/2) = -15/16$, and $f(3) = 4$, the function f has zeros in the intervals $2 < x < 5/2$ and $5/2 < x < 3$. Moreover, since $f(-3) = 52$, $f(-2) = -6$, $f(-1) = -4$, $f(0) = 10$, f also has zeros in the intervals $-3 < x < -2$ and $-1 < x < 0$.

57. $f(x) = x^3 - x^2 - 3x + 1$

$f'(x) = 3x^2 - 2x - 3$

If $f'(x) = 0$, $x = \frac{1 \pm \sqrt{10}}{3}$; hence relative maximum and minimum points occur at points $(-.7, 2.3)$ and $(1.4, -2.4)$ approximately. The graph has a point of inflection at $x = \frac{1}{3}$.



Graph of $f: x \longrightarrow x^3 - x^2 - 3x + 1$

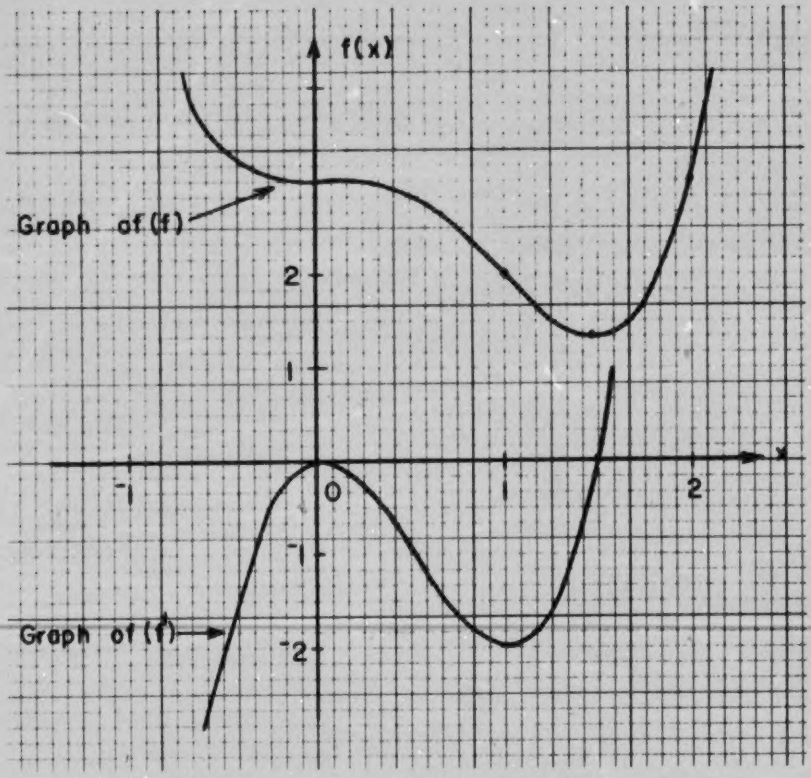
The smallest root of $x^3 - x^2 - 3x + 1 = 0$ is in the interval $-2 < x < -1$, since $f(-2) = -5$ and $f(-1) = 2$. By Newton's method we obtain the desired root as -1.48, correct to two decimals.

58. $f(x) = x^4 - 2x^3 + 3$

$f'(x) = 4x^3 - 6x^2 = g(x)$

Thus $g'(x) = 12x^2 - 12x = 12x(x - 1)$; $g'(x) = 0$ if and only if $x = 0$ or $x = 1$.

Points $(0, 0)$, $(1, -2)$, and $(1/2, -1)$ are relative maximum, minimum points, and point of inflection, respectively, of the graph of g.



With respect to the function f, the point (0, 3) is a point of inflection, (1, 2) is a point of inflection, and (1.5, 1.3125) is a relative minimum point. This is clear from $f(x) = 2 - 2(x - 1) + 2(x - 1)^3 + (x - 1)^4$ and

$$f(x) = 1.3125 + 0(x-3/2) + 9/2(x-3/2)^2 + 4(x-3/2)^3 + (x-3/2)^4.$$

59. If $f'(x_1) = 0$, we must use another estimate, since division by zero is not defined. Geometrically, of course, the tangent is horizontal and there is no intersection with the x-axis.

60. The related slope function $x \longrightarrow \frac{|x|}{x} = \begin{Bmatrix} 1 \text{ for } x > 0 \\ -1 \text{ for } x < 0 \end{Bmatrix}$ is defined only for $x \neq 0$, and hence cannot be used to find the minimum value since division by 0 is impossible. At $x = 0$, the absolute value function $x \longrightarrow |x|$ attains a minimum value of 0.

Illustrative Test Questions for Chapter 3

The following test items are merely suggestive. The order of the questions is based upon the order in which the material tested appears in the text; questions 13 - 16 require *more working time* than 1 - 12.

For a full period test (40 to 50 minutes) a selection of items should be made. Thus, for example, a full period test might consist of questions 2, 3a, 5, 12, 13.

1. Find the equation of the tangent T to the graph G of $g: x \longrightarrow 2 + 3x + x^5$ at its point P of intersection with the y-axis. Near P does the graph lie above T, below T, or does it cross over T?
2. If it is desired that near $P(0, 2)$ the graph of $f: x \longrightarrow 2 + 7x - 2x^2$ lie between the straight lines $y = 2 + 7.001x$ and $y = 2 + 6.999x$, what values may x assume?
3. For each of the following, draw the tangent to the graph at its point of intersection with the y-axis and sketch the shape of the graph in the vicinity of this point.
   a) $x \longrightarrow 3 - 2x + x^3$
   b) $x \longrightarrow 3 + 2x - x^2$
4. Write the expansion of $g(x)$ in powers of $x - h$ and determine the equation of the tangent to the graph of g at $\left(h, g(h)\right)$.
   a) $g: x \longrightarrow 2x^3 - 6x^2 + 1$
   b) $g: x \longrightarrow x^4 - x^2$
5. Given $f: x \longrightarrow 3 - 4x + x^3 - x^4$. Expand $f(x)$ in powers of $(x + 2)$.
6. Find the equation of the tangent to the graph of $x \longrightarrow 2 + 5x - x^2 + x^3$ at $(2, 16)$.
7. Find the slope of the tangent to the graph of $x \longrightarrow x^3 + px^2 + qx + s$ at the point where $x = t$.

8. Characterize each critical point of the function

$$f: x \longrightarrow x^4 - 4x^3.$$

9. Find an equation of the tangent to the graph of

$$f: x \longrightarrow x^3 - 3x + 6$$

at its point of inflection.

10. Characterize the point (1, 0) on the graph of $x \longrightarrow (x - 1)^3$ as a relative maximum point, relative minimum point, point of inflection, or none of these. Sketch the graph and its tangent near the point (1, 0).

11. Given the polynomial

$$f: x \longrightarrow a_0 + a_1x + a_2x^2 + \dots + a_nx^n,$$

find the associated slope function.

12. Divide 30 into two parts such that the sum of the squares of these parts is a minimum.

13. Find and identify the character of each critical point of

$$f: x \longrightarrow (x + 2)^2(x - 2).$$

14. Find correct to two decimals the real root of $x^5 + 9 = 0$.

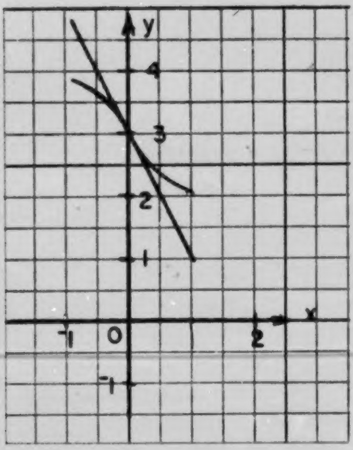
15. Find correct to two decimals the positive zero of

$$f: x \longrightarrow x^3 + 3x^2 - 6x - 3.$$

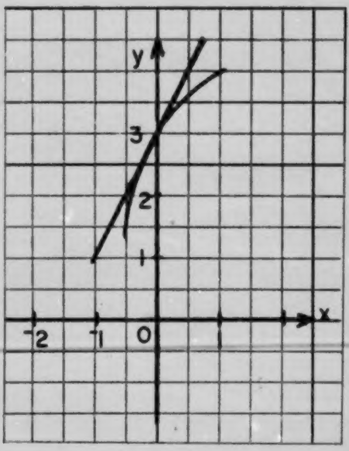
16. Show that there are exactly two points on the graph of the function $f: x \longrightarrow x^2 - 4$ that are at a minimum distance from the point (0, -2).

Answers to Illustrative Test Questions for Chapter 3

1. Tangent: $y = 2 + 3x$
   Graph G lies below T at the left of $P(0, 2)$ and above T at the right of P.
2. $|2x| < 0.001$ or $|x| < 0.0005$.
3. a)



b)



4. a) $g(x) = (2h^3-6h^2+1) + (6h^2-12h)(x-h) + (6h-6)(x-h)^2 + 2(x-h)^3$

   T: $y = (2h^3 - 6h^2 + 1) + (6h^2 - 12h)(x - h)$

   b) $g(x) = (h^4-h^2) + (4h^3-2h)(x-h) + (6h^2-1)(x-h)^2 + 4h(x-h)^3 + (x - h)^4$

   T: $y = (h^4 - h^2) + (4h^3 - 2h)(x - h)$
5. $f(x) = -13 + 40(x + 2) - 30(x + 2)^2 + 9(x + 2)^3 - (x + 2)^4$
6. $x \longrightarrow 2 + 5x - x^2 + x^3 = 16 + 13(x - 2) + 5(x - 2)^2 + (x - 2)^3$

   T: $y = 16 + 13(x - 2)$ at $(2, 16)$.
7. $f(x) = x^3 + px^2 + qx + s$

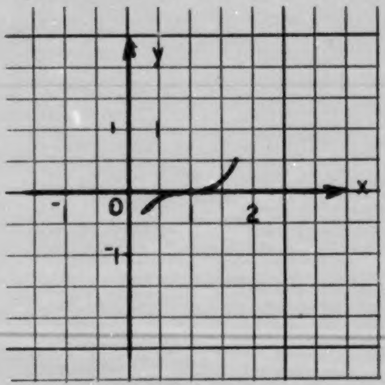
   $f'(x) = 3x^2 + 2px + q$

   $f'(t) = 3t^2 + 2pt + q = m.$

8. Point $(0, 0)$ is point of inflection.
Point $(3, -27)$ is minimum point.

9. $f(x) = x^3 - 3x + 6$
$f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1)$
Point of inflection at $\frac{-1 + 1}{2} = 0$
$y = -3x + 6$ is tangent line at $(0, 6)$.

10. Point $(1, 0)$ is a point of inflection; the x-axis is tangent line.



11. $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$
$f': x \longrightarrow a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1}$

12. $S = x^2 + (30 - x)^2 = 2x^2 - 60x + 900 = f(x)$
$f'(x) = 4x - 60$; If $f'(x) = 0$, $x = 15$.
Numbers are 15, 15.

13. $f(x) = (x + 2)^2(x - 2) = x^3 + 2x^2 - 4x - 8$
$f'(x) = 3x^2 + 4x - 4 = (x + 2)(3x - 2)$
$(-2, 0)$ is relative maximum point;
$(\frac{2}{3}, -9\frac{13}{27})$ is relative minimum point.

14. -1.55

15. 1.67

16. If distance = D, then $D^2 = x^2 + \left(x^2 - 4 - (-2)\right)^2$
$f: x \longrightarrow x^2 + (x^2 - 2)^2 = x^4 - 3x^2 + 4$
$f'(x) = 4x^3 - 6x = 2x(2x^2 - 3)$

Critical points at $x = 0$, $x = \pm\sqrt{3/2}$.

The expansion of $f(x)$ in powers of $(x - \sqrt{3/2})$,

$$f(x) = \frac{7}{4} + 6\left(x - \sqrt{\frac{3}{2}}\right)^2 + 4\sqrt{\frac{3}{2}}\left(x - \sqrt{\frac{3}{2}}\right)^3 + \left(x - \sqrt{\frac{3}{2}}\right)^4,$$

shows that f has a relative minimum value at $x = \sqrt{3/2}$.
Similarly,

$$f(x) = \frac{7}{4} + 6\left(x + \sqrt{\frac{3}{2}}\right)^2 - 4\sqrt{\frac{3}{2}}\left(x + \sqrt{\frac{3}{2}}\right)^3 + \left(x + \sqrt{\frac{3}{2}}\right)^4$$

shows that f has a relative minimum value at $x = -\sqrt{3/2}$ also. In each case the distance is a minimum, that is, $D = \frac{1}{2}\sqrt{7}$.

Special Questions

The following questions involve some of the ideas of the preceding chapters in a novel or challenging way. They should not be considered an essential part of the course, but they may give some amusement and added insight to able students.

1. Show that $x^2 + 1$ is a factor of

$$f(x) = x^4 + x^3 + 3x^2 + x + 2$$

by showing that $f(i) = 0$ and $f(-i) = 0$.

2. For what values of k will $f: x \longrightarrow x^3 - 3x + k$ have a zero of multiplicity two?

3. Find the solution set of the equation

$$|x|^3 - 2|x|^2 - 5|x| + 6 = 0.$$

4. If the function $f: x \longrightarrow y$ is completely specified by the table, and if g is an inverse of f, state the domain of fg and the domain of gf. Are fg and gf the same function?

x	1	2	3
y	2	3	4

5. a) Sketch a graph of $g: x \longrightarrow 4 + |x^2 - 4|$

   b) What is the slope of the graph at $x = 0$, $x = 1$, $x = 3$, $x = 2$?

   c) What is the minimum value of g?

6. The symbol $[x]$ is used to denote the greatest integer not exceeding x, that is, $[x]$ is an integer and $x - 1 < [x] \leq x$. Thus, $[3] = [\pi] = 3$, and $[-4] = [-\pi] = -4$.

   a) Sketch the graphs of $y = [x]$ and $y = [-x]$ for $-2 \leq x \leq 2$.

   b) When does $[-x] = -[x]$?

   c) Using this notation, find expressions for the least integer which x does not exceed and the greatest integer which x exceeds.

d) What is the slope function associated with the function $f: x \longrightarrow [x]$ for non-integral $x > 1$?

7. If the coefficients a_1 are real numbers, how many roots has $a_2|x|^2 + a_1|x| + a_0 = 0$?

Answers to Special Questions

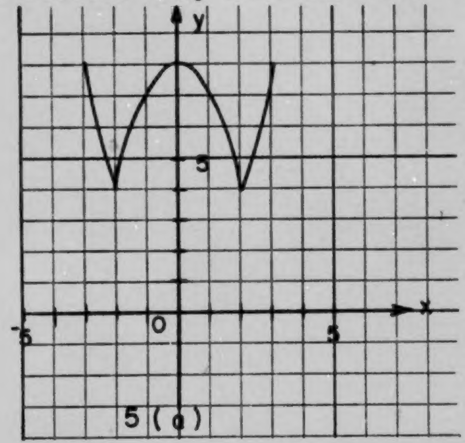
1. $f(i) = 1 - i - 3 + i + 2 = 0$
$f(-i) = 1 + i - 3 - i + 2 = 0$

2. $f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$ and the only possible zeros of f of multiplicity two are therefore 1 and -1. In the former case, $f(1) = 1 - 3 + k = 0$ and $k = 2$; in the latter case, $k = -2$.

3. The given equation implies $|x| = -2, 1$ or 3. The first alternative is not possible for any x; the second and third yield the solution set $\{1, -1, 3, -3\}$.

4. Domain of fg = domain of $g = \{2, 3, 4\}$.
Domain of **gf** = domain of $f = \{1, 2, 3\}$. These functions are not the same because they have different domains.

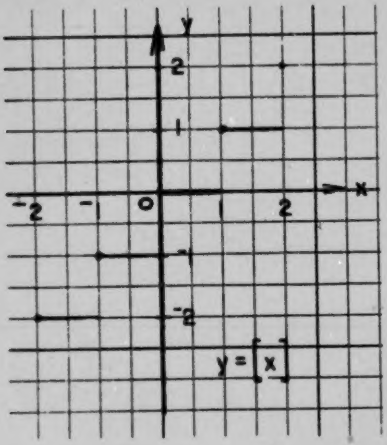
5. a)

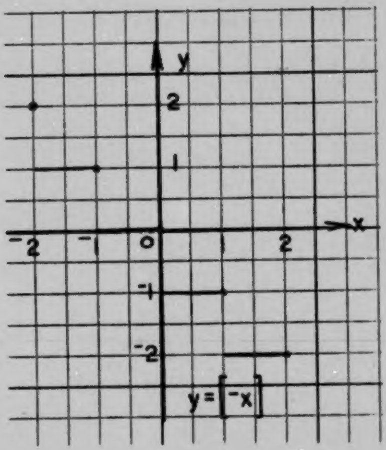


b) The slope is 0 at $x = 0$, -2 at $x = 1$, 6 at $x = 3$, and no slope is defined at $x = 2$.

c) 4.

6. a)





b) When x is an integer.

c) $-[-x]$; $-[-x] -1$.

d) $f': x \longrightarrow 0$.

7. Four or fewer. For example,

if $(a_2, a_1, a_0) = (1, -3, 2)$, roots are $\pm 1, \pm 2$;
$(1, -1, 0)$, $0, \pm 1$;
$(1, 1, -2)$, ± 1;
$(1, 1, 0)$, 0;
$(1, 2, 1)$, no roots.

Chapter 4

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Introduction

The purpose of this chapter is to study the properties of the exponential functions $f: x \longrightarrow a^x (a > 0)$ and their inverses, the logarithmic functions.

It is assumed that the student is familiar with the laws of exponents, in particular with

$$a^{r+s} = a^r a^s$$

and

$$(a^r)^s = a^{rs}$$

where r and s are rational numbers. Nevertheless, these matters are reviewed in the first two sections in connection with a concrete problem -- the growth of a colony of bacteria. In Section 4-3, a^x is given a meaning when x is irrational. An alternate approach to exponential functions is given in an Appendix. (See Section 4-15). This alternative introduces and solves the functional equation $f(x + y) = f(x)f(y)$. We believe that this development will be very illuminating for superior students.

It is shown in Section 4-4 that we can write an arbitrary number a as a power of 2 and that it is therefore sufficient to treat the single exponential function $x \longrightarrow 2^x$.

Preparatory to the treatment of tangents to the graph of $x \longrightarrow 2^x$, it is made plausible (Section 4-5) that the graph is concave upward everywhere. A strict proof would require an argument from continuity.

We are now ready to apply the "wedge" method of Chapter 3. As in the case of polynomial graphs, it is convenient to begin with the point on the y-axis. We do not prove rigorously that the wedges can be made arbitrarily narrow. However, in the text and exercises, they are chosen sufficiently narrow to make it plausible that there is a unique linear approximation to the graph at (0,1) with a slope $k \approx 0.693$ and that therefore

$$2^x \approx 1 + kx \quad \text{for } |x| \text{ small.}$$

At this stage it is expedient to take leave of the base 2 and adopt the base e for which the linear approximation is as simple as possible, namely, $e^x \approx 1 + x$ for $|x|$ small. The slope of the graph at (0,1) is then 1. More generally the slope of the graph of $f: x \rightarrow e^x$ at (h, e^h) is the ordinate of the point. That is, the slope function f' is identical with f. The base e is easily seen to be $2^{1/k} \approx 2.718$.

The applications discussed in 4-8 are of three types: to radioactive decay, to compound interest, and to cooling. It is expected that at least the first type will be included in view of the current interest in radioactivity. It has the advantage that only powers of 2 need to be involved.

We prepare for the introduction of logarithmic functions by continuing (see Section 4-9) the discussion of inverse functions begun in Section 1-6. It may be desirable to review the earlier material at this time. We believe that the logarithmic functions take on added meaning in the setting of the general principles which govern inversion.

After discussing logarithms and change of base, methods are given for computing e^x and $\ln x$. In this text, we have made no use of infinite processes. Accordingly, we avoid the language of infinite series. Instead we approximate e^x for $|x|$ small by substituting x in an appropriate polynomial. Similarly, we use polynomials to estimate $\ln x$ for x near 1. These matters are treated in general terms without attempting to prove every statement. There are, however, further discussions in the Appendices (Sections 4-16, 4-17 and 4-18).

The final brief section (4-13) on the history of logarithms and some of the included references may, if desired, be used as the basis of an assigned paper.

4-1. Introduction. Pages 145-149.

It may be appropriate at this point to review the basic meanings of rational exponents and work a few exercises of the following kind:

1. Find a simpler name for

   a) $2^3 \cdot 2^2$ b) $2^3 / 2^2$ c) $2^2 \cdot 2^3$ d) $4^{3/2}$

   e) $4^{-3/2}$ f) $8^{2/3}$ g) $8^{-2/3}$ h) $(-8)^{-2/3}$

2. Find a simpler name for

   a) $a^m \cdot a^n$ b) a^m / a^n c) $(a^m)^n$

   d) a^{-n} e) $(ab)^m$ f) $(\frac{a}{b})^m$

   g) What restrictions do we have on a, b, m, n? We avoid values that yield a zero denominator or an expression of the form 0^0 .

In the special case of bacteria, the principle of growth shows that the number N of bacteria doubles in a fixed time period, not necessarily in a day. Actually, the number present 24 hours after a count of N_0 is made might be $\frac{3}{2} N_0$, $5N_0$, ..., or more generally $a \cdot N_0$. In such a case the number present at the end of 48 hours, or 2 days, is $a \cdot (a \cdot N_0) = a^2 N_0$; hence, at the end of n days, the number present $N(n)$ is $a^n \cdot N_0$. This is the basis for the formula used in the solution of Exercise 7 in this section.

Answers to Exercises 4-1. Pages 149-150.

1. If $2^0 \cdot 2^r = 2^{0+r} = 2^r$, then

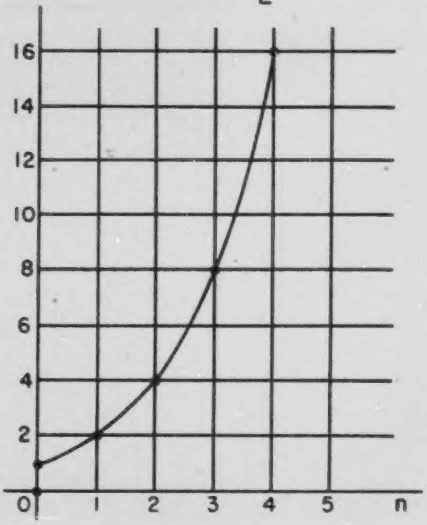
$$\frac{2^{0+r}}{2^r} = \frac{2^r}{2^r} = 1; \text{ and since from (1)}$$

$$\frac{2^{0+r}}{2^r} = 2^0, \ 2^0 \text{ must be defined to be 1.}$$

2. If $2^{-r}\cdot 2^{r} = 2^{0} = 1$, then by division, $2^{-r} = \frac{1}{2^{r}}$, since $2^{r} \neq 0$.

3. Let $N(n) = 10^{6}(2^{n})$

n	N(n) in millions
0	1
1	2
2	4
3	8
4	16



4. $N(n) = 10^{6}(2^{n})$ so that

$$\frac{N(n+5)}{N(n+2)} = \frac{10^{6}(2^{n+5})}{10^{6}(2^{n+2})} = 2^{3} = 8$$

5. $$\frac{N(n+7)}{N(n-3)} = \frac{2^{(n+7)}}{2^{(n-3)}} = 2^{10} = 1024$$

6. If $N = N(n+100) = 10^{6}(2^{100})$, then

$\frac{N}{4} = \frac{10^{6}(2^{100})}{2^{2}} = 10^{6}(2^{98}) = N(98)$. Thus, after 98 days there were $\frac{N}{4}$ present.

7. $$N(n) = N_0\cdot a^{n}$$

$$200{,}000 = N(n+3) = N_0(a^{n+3})$$

and

$$1{,}600{,}000 = N(n+\tfrac{9}{2}) = N_0(a^{n+\frac{9}{2}})$$

Thus

$$\frac{N(n+\frac{9}{2})}{N(n+3)} = a^{\frac{3}{2}} = 8, \text{ and } a = 4;$$

hence

$N(n) = N_0(4^{n})$, $N(3) = N_0(4^{3}) = 200{,}000$, and $N_0 = 200{,}000(4^{-3})$.

The formula for $N(n)$ becomes

$$N(n) = 200{,}000 \cdot 4^{n-3}.$$

(a) If $n = 5$, $N(5) = 200{,}000 \cdot 4^{5-3} = 3{,}200{,}000$

(b) If $n = 3/2$, $N(3/2) = 200{,}000 \cdot 4^{(3/2)-3}$

$$= 200{,}000 \cdot 4^{-3/2} = 200{,}000 \cdot 1/8$$

$$= 25{,}000.$$

(c) If $N(n) = 800{,}000$, then $200{,}000 \cdot 4^{n-3} = 800{,}000$ and $4^{n-3} = 4^1$. Therefore, $n - 3 = 1$ and $n = 4$.

4-2. Rational Powers of Positive Real Numbers. Pages 150-156.

The basic principle of the law of growth may be extended to the general case. If the number of bacteria present at the end of one day is $a \cdot N_0$, then the number present at the end of t days is $a^t \cdot N_0$. The multiplying factor a^t does not depend upon the time when the initial count N_0 is made. This principle is illustrated in Exercise 7 in Section 4-1, since

$$\frac{N(n + 9/2)}{N(n + 3)} = \frac{1{,}600{,}000}{200{,}000} = 8$$

and

$$\frac{N(n + 3)}{N(n + 3/2)} = \frac{200{,}000}{25{,}000} = 8.$$

Exercises 4-2a are introduced as an application of the basic laws of exponents (3) and (4). It is expected that this will constitute a minimum review; however, the time required for this work will necessarily vary from class to class. Since the development which follows does not demand extensive manipulative skills, it is possible to provide for review work, if necessary, while continuing with the discussion of the text.

It might be well for the student to obtain some rational powers of 2 before resorting to Table 4-2. In this way he will develop an intuitive notion of the increasing nature of the function $f: x \longrightarrow 2^x$, and the table will be more meaningful to him. The rational powers of 2 that are most readily computed are

$$2^{.5},\ 2^{.25},\ 2^{.75},\ 2^{.125},\ 2^{.625},\ 2^{.875},$$

which are obtained by taking successive square roots of 2 and appropriate products. Thus, $2^{.75} = 2^{.5} \cdot 2^{.25}$. This could have been obtained, also, by finding the square root of $\sqrt{8}$. Thus, $2^{3/4} = 8^{1/4}$. If these values are obtained and listed in order, we get approximately,

r	2^r
0.000	1.000
.125	1.091
.250	1.189
.375	1.296
.500	1.414
.625	1.542
.750	1.682
.875	1.834
1.000	2.000

Intermediate values of 2^r may be obtained by linear interpolation within a very small error. Thus, using the familiar interpolation technique, we get from this table

$$2^{.2} = 1.150 \text{ instead of } 1.149,$$
$$2^{.8} = 1.742 \text{ instead of } 1.741.$$

It might be helpful and a time-saver to have a class-chart of the table of values of 2^r as given on page 235.

The treatment of exponential functions is introduced via the use of 2 as the base, that is $x \longrightarrow 2^x$. For this reason we feel that appropriate significant experiences with a table of powers of 2 can be used as an intuitive basis for the study of $x \longrightarrow 2^x$, for x real. Thus, it becomes plausible that there are numbers which might be used to fill in the gaps in the table. In point of fact, the student should be led to observe that every positive real number r could be written as an entry in the table, that is, as a power of 2. In this way, the use of the table leads to the discussion of Section 4-3.

The computation involved in Exercises 4-2b can be simplified by devices such as rounding the entries to 2 or 3 decimal-place accuracy, and rationalizing the denominator (See Example 2). The objective of the section is not realized if straight-line interpolation is used to obtain entries not in the table; it is intended that the laws of exponents (3) and (4) should be the basis for the method used.

Answers to Exercises 4-2a. Page 153.

1. a) If both m and n are positive integers, $a^m \cdot a^n =$

$$\underbrace{a \cdot a \cdot \ldots \cdot a}_{m \text{ factors}} \cdot \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}} = \underbrace{a \cdot a \cdot \ldots \cdot a}_{m + n \text{ factors}} = a^{m+n}$$

If $m = 0$, then we know that $a^0 = 1$. Hence,
$a^m \cdot a^n = a^0 \cdot a^n = 1 \cdot a^n = a^n = a^{0+n} = a^{m+n}$.

A similar argument holds if $n = 0$. Thus, the rule is established for $m, n = 0, 1, 2, \ldots$.

b) Now suppose $n < 0$, $m \geq 0$. Then

$$a^m \cdot a^n = a^m \cdot \frac{1}{a^{-n}} = \frac{a^m}{a^{-n}} .$$

(1) If $m \geq -n$, let $m = -n + k$ where $k = 0, 1, 2, \ldots$. Then

$$\frac{a^m}{a^{-n}} = \frac{a^{-n+k}}{a^{-n}} = \frac{a^{-n} \cdot a^k}{a^{-n}} \quad \text{from (2)}$$

$$= a^k = a^{m+n}, \text{ so } a^m \cdot a^n = a^{m+n}$$

for this case.

(2) If $m < -n$ (m need not be 0), then let $m + k = -n$ where $k = 1, 2, 3, \ldots$. Hence

$$\frac{a^m}{a^{-n}} = \frac{a^m}{a^{m+k}} = \frac{a^m}{a^m \cdot a^k} = \frac{1}{a^k} = a^{-k} = a^{m+n}$$

Once again $a^m \cdot a^n = a^{m+n}$, completing all the possibilities.

2. a) Suppose $m, n = 0, 1, 2, 3, \ldots$.

If $m = 0$, then $(a^m)^n = (a^0)^n = 1^n = 1 = a^0$

$$= a^{0 \cdot n} = a^{mn}.$$

If $n = 0$, then $(a^m)^n = (a^m)^0 = 1 = a^0 = a^{0 \cdot m} = a^{mn}$.

Now suppose $m \neq 0$, $n \neq 0$. Then $(a^m)^n$ means that a^m is used as a factor n times:

$$(a^m)^n = \underbrace{a^m \cdot a^m \cdot \ \ldots \ \cdot a^m}_{n \text{ factors of } a^m}.$$

But a^m itself is the product when a is taken as a factor m times, $a^m = \underbrace{a \cdot a \cdot \ \ldots \cdot a}_{m \text{ factors}}$. Replacing "$a^m$" by its expanded form, "$a \cdot a \cdot \ \ldots \ \cdot a$" in $(a^m)^n$ we see that a is used as a factor mn times so that

$$(a^m)^n = a^{mn}.$$

We have established the rule $(a^m)^n = a^{mn}$ for $m, n = 0, 1, 2, 3, \ldots$. In fact, if m or $n = 0$, the other need not be restricted to the non-negative integers. The proof goes through for this more general case.

b) Now suppose $m = -1, -2, \ldots; \; n = 1, 2, 3, \ldots$.

Then $$(a^m)^n = \left(\frac{1}{a^{-m}}\right)^n = \underbrace{\frac{1}{a^{-m}} \cdot \frac{1}{a^{-m}} \cdot \ldots \cdot \frac{1}{a^{-m}}}_{n \text{ factors}}$$

$$= \frac{1}{(a^{-m})^n} = \frac{1}{a^{(-m)n}} = \frac{1}{a^{-mn}} = a^{mn} .$$

c) Now suppose, $m = 1, 2, \ldots; \; n = -1, -2, -3, \ldots$

Then $$(a^m)^n = \frac{1}{(a^m)^{-n}} = \frac{1}{a^{m(-n)}} = \frac{1}{a^{-mn}} = a^{mn}.$$

d) Finally, suppose $m = -1, -2, \ldots; \; n = -1, -2, \ldots$.

Then $$(a^m)^n = \left(\frac{1}{a^{-m}}\right)^n = \frac{1}{\left(\frac{1}{a^{-m}}\right)^{-n}} = \frac{1}{\frac{1}{(a^{-m})^{-n}}}$$

$$= \frac{1}{\frac{1}{a^{(-m)(-n)}}} = \frac{1}{\frac{1}{a^{mn}}} = a^{mn}.$$

This completes the proof that $(a^m)^n = a^{mn}$ for integral values of m and n.

3. $1000(8^{-2/3}) = 1000 \cdot \frac{1}{8^{2/3}} = 1000 \cdot \frac{1}{2^2} = 250$

$3(\frac{9}{4})^{-3/2} = 3(\frac{3}{2})^{-3} = 3(\frac{2}{3})^3 = 3 \cdot \frac{8}{27} = \frac{8}{9}$.

4. $(4^{5/2})(8^{-1}) = (2^2)^{5/2}(2^3)^{-1} = 2^5 \cdot 2^{-3} = 2^2$

$(\frac{1}{2})^{-4/3} = (2^{-1})^{-4/3} = 2^{4/3}$

$(2^{-2/9})^9 = 2^{-2}$.

Since $x \longrightarrow 2^x$ is an increasing function as x increases, in order of decreasing value from the left we have

$$2^2,\ 2^{4/3},\ 2^{2/3},\ 2^{-2},\ 2^{-3} \text{ or}$$

$$(4^{5/2})(8^{-1}),\ (\tfrac{1}{2})^{-4/3},\ 2^{2/3},\ (2^{-2/9})^9,\ 2^{-3}.$$

5. $2^{2.7} = 2^2 \cdot 2^{.7} = 4 \cdot 2^{7/10} = 4 \cdot \sqrt[10]{2^7} = 4 \cdot \sqrt[10]{128}$

6. a) If $8^m = (2^3)^2$, then $8^m = 8^2$ and $m = 2$.

   b) If $8^m = 2^{(3^2)}$, then $2^{3m} = 2^9$ and $m = 3$.

7. a) If $2^{(4^5)} = 16^m$, then $2^{(4^5)} = (2^4)^m = 2^{4m}$ and $4^5 = 4m$ so that $m = 4^4 = 256$.

   b) If $(2^4)^5 = 16^m$, then $2^{20} = 2^{4m}$ and $20 = 4m$, so that $m = 5$.

8. $\dfrac{2^h + 2^{h+2b}}{2} = \dfrac{2^h + 2^h \cdot 2^{2b}}{2} = \dfrac{2^h(1 + 2^{2b})}{2}$

Answers to Exercises 4-2b. Page 156.

1. a) $2^{5/4} = 2^{1.25} = 2 \cdot 2^{.25} \approx 2(1.189) = 2.378$

   b) $2^{5/4} = 2\sqrt{\sqrt{2}} \approx 2\ \sqrt{1.414} \approx 2(1.189) = 2.378$

2. a) $2^{1.15} = 2 \cdot 2^{.15} \approx 2(1.110) = 2.220$

   b) $2^{2.65} = 2^2 \cdot 2^{.65} \approx 4(1.569) = 6.276$

   c) $2^{0.58} = 2^{.55} \cdot 2^{.03} \approx (1.464)(1.021) \approx 1.495$

   d) $2^{-0.72} = 2^{-1+.28} = \frac{1}{2}(2^{.25})(2^{.03})$

      $\approx .5(1.189)(1.021) \approx .607$

3. a) $8^{.84} = (2^3)^{.84} = 2^{2.52} = 2^2 \cdot 2^{.5} \cdot 2^{.02}$

$\approx 4(1.414)(1.014) \approx 5.735$

b) $0.25^{-0.63} = (2^{-2})^{-0.63} = 2^{1.26} = 2^1 \cdot 2^{.25} \cdot 2^{.01}$

$\approx 2(1.189)(1.007) \approx 2.395.$

4.

r	r^2
-4.0	.0625
-3.6	.0825
-3.2	.109
-2.8	.144
-2.4	.189
-2.0	.250
-1.6	.330
-1.2	.435
1.4	2.639
1.8	3.482
2.2	4.595
2.6	6.063
3.0	8.000

4-3. Arbitrary Real Exponents. Pages 157-162.

In this section we are faced with the problem of filling in values of 2^x for irrational values of x. The condition or requirement that $x \longrightarrow 2^x$ be an increasing function, comes to our rescue. With its aid $f: r \longrightarrow 2^r$, defined for all rational numbers r, can be extended in a unique way to a function $f: x \longrightarrow 2^x$ defined for all real numbers x. The illustrative example chosen, $2^{\sqrt{2}}$, might have been 2^{π} or any irrational power of 2. The "pinching down" process makes use, in a concrete sense, of the notion of nested intervals in order to give meaning to 2^x, x irrational. The development is presented at an intuitive level.

Exercises involving the use of the graph of $f: x \longrightarrow 2^x$ are provided to give the student some acquaintance with the special characteristics of f. He should note that the domain of f is the set of all real numbers and the range, the set of all positive real numbers; furthermore, f increases steadily for all x. The graph shows clearly that f is a one-to-one function.

Answers to Exercises 4-3. Page 163.

2. A comparison should reveal a difference of not more than 0.2.

3. a) $2^{\sqrt{3}} \approx 2^{1.73} \approx 3.3$
   b) $2^{\pi} \approx 2^{3.14} \approx 8.8$
   c) $2^{-\pi/4} \approx 2^{-.79} \approx .6$

4. No, there is no real number x such that $2^x = 0$. We give two arguments to support this statement. The second is very neat, but more sophisticated.

   a) There is no such real number $x \geq 0$ since $f: x \longrightarrow 2^x$ is an increasing function and $2^0 = 1$ so that $f(x) \geq 1$ for $x \geq 0$.

   If $x < 0$ we set $t = -x$; hence, $2^x = \frac{1}{2^t}$. Since there is no number 2^t such that $\frac{1}{2^t} = 0$, there is no real number x such that $2^x = 0$.

   b) If x is a real number such that $2^x = 0$, and if y is any real number whatever, then $y - x$ is a real number, 2^{y-x} is therefore defined, and we have

   $$2^{y-x} \cdot 2^x = 2^{y-x} \cdot 0 = 0,$$

   but

   $$2^{y-x} \cdot 2^x = 2^y,$$

therefore

$$2^y = 0$$

for every real number y. But $2^1 = 2$, so this is impossible.

5. a) If $2^x = 6$, $x \approx 2.6$ d) If $2^x = 3$, $x \approx 1.6$
   b) If $2^x = .4$, $x \approx -1.3$ e) If $2^x = 2.7$, $x \approx 1.4$
   c) If $2^x = 3.8$, $x \approx 1.9$

4-4. Powers of the Base a as Powers of 2. Pages 163-166.

In this section we bridge the gap between the study of $x \longrightarrow 2^x$ and $x \longrightarrow a^x$ for any base $a > 0$. Using the basic properties of $f: x \longrightarrow 2^x$, we proceed to show how any real positive number might be considered as an entry in the table of powers of 2, or as the ordinate of a point on the graph of f.

The illustrative examples have been selected and arranged according to difficulty; the laws of exponents (3) and (4) given in Section 4-2 provide the basis for the method used in the solution.

The graph in Figure 4-3c should be used in conjunction with Table 4-2 to check answers to Exercises 4-4. The graph may be used in lieu of the table, although the results will not be as accurate.

Answers to Exercises 4-4. Page 166.

1. We first write $3.4 = 2(1.7)$. In order to express 1.7 as a power of 2 we may,
   a) use the graph and read $2^{.77} \approx 1.7$,
   b) interpolate in Table 4-2 between the entries 1.68179 and 1.74110, obtaining $x \approx .77$,
   c) if we do not object to the calculation involved, divide 1.7 by 1.682 (since entry $1.68179 \approx 1.682$) obtaining $1.7 \approx (1.682)(1.0107)$. Using the table again we note

that

$$2^{.01} < 1.0107 < 2^{.02};$$

hence $\quad 1.7 \approx (2^{.75})(2^{.02}) = 2^{.77}$.

Getting back to the given exercise we may now express $3.4 = 2(1.7)$ as $2^1(2^{.77}) = 2^{1.77}$ (approximately). We should note that the graph gives a satisfactory approximation.

2. We write $2.64 = 2^1(1.32)$; then, since

$$2^{.40} \approx 1.32,$$

$$2.64 \approx 2^1(2^{.40}) = 2^{1.40}.$$

Thus, $(2.64)^{0.3} \approx (2^{1.40})^{0.3} = 2^{0.42} = (2^{0.40})(2^{0.02})$.

From Table 4-2, we have

$$2^{0.4} \approx 1.320, \; 2^{0.02} \approx 1.014;$$

hence $(2.64)^{0.3} \approx (2^{0.4})(2^{0.02}) \approx (1.32)(1.014) \approx 1.34$.

Alternatively, the graph of $x \longrightarrow 2^x$ may be used twice: first to see that $2.64 \approx 2^{1.4}$; then to find

$$(2^{1.4})^{0.3} = 2^{0.42}.$$

Using the graph, we obtain the result

$$2^{0.42} \approx 1.3$$

3. We first write $6.276 = 4(1.569)$. The Table or graph gives

$$1.569 \approx 2^{0.65}$$

so that

$$6.276 = 2^2(1.569) \approx 2^2(2^{0.65}) = 2^{2.65}.$$

Then

$$(6.276)^{-0.6} \approx (2^{2.65})^{-0.6} = 2^{-1.59}$$

$$= 2^{-2+.41} \quad = 2^{-2}(2^{.41})$$

$$\approx \tfrac{1}{4}(1.33) \quad \approx 0.33.$$

Alternatively, to use the graph we round

$6.276 \approx 6.28$ and read off $2^{2.65} \approx 6.28$.

Then since

$$(6.28)^{-0.6} \approx (2^{2.65})^{-0.6} = 2^{-1.59} \approx 2^{-1.6},$$

we refer to the graph again and read off

$$2^{-1.6} \approx 0.35.$$

The result is satisfactory, and quickly obtained.

4. Since $5.2 = 4(1.3)$ and $1.3 \approx 2^{.38}$

$$5.2 \approx 2^2(2^{.38}) = 2^{2.38}$$

Thus,

$$(5.2)^{2.6} \approx (2^{2.38})^{2.6} = 2^{6.188} \approx 2^6(2^{0.19}).$$

But

$$2^{0.19} \approx 1.14,$$

hence

$$2^6(2^{0.19}) \approx 2^6(1.14) = 72.96.$$

We have

$$(5.2)^{2.6} \approx 73.$$

5. Prove that if $0 < a < 1$ and $v > u$, then $a^v < a^u$.

Proof: Let $b = 1/a$. Then $b > 1$ and $b^v > b^u$, if $v > u$. Dividing by $b^v b^u$,

$$\frac{1}{b^u} > \frac{1}{b^v}$$

$$\left(\frac{1}{b}\right)^u > \left(\frac{1}{b}\right)^v$$

and finally $$a^u > a^v.$$

4-5. A Property of the Graph of $x \longrightarrow 2^x$. Pages 167-169.

The proof in this section (but not the result) may be omitted if you are pressed for time. The topic is, however, of considerable importance in modern mathematics. In more advanced courses, the function $x \longrightarrow 2^x$ and its graph are said to be convex (meaning, convex downward). We have adopted the more familiar expression "concave upward".

The theorem on the relation of arithmetic and geometric means is an interesting and important one. In Exercises 4 and *5, we have suggested a proof of the extension of the theorem to three and four numbers. This beautiful proof, due to Cauchy, is not for everyone.

Answers to Exercises 4-5. Pages 169-170.

1. Given $P(.05, 2^{.05})$ and $Q(.25, 2^{.25})$

   Point M has coordinates (.15, 1.1122), approximately, since

   $$\tfrac{1}{2}(2^{.05} + 2^{.25}) \approx \tfrac{1}{2}(1.03526 + 1.18921) \approx 1.11223.$$

   Point R has coordinates

   $$(.15, 2^{.15}) \approx (.15, 1.1096).$$

   The ordinate of M is greater than the ordinate of R, that is $1.1122 > 1.1096$.

2. M_1 has coordinates (.10, 1.0724) since

   $$\tfrac{1}{2}(1.03526 + 1.10957) \approx 1.07242.$$

   The corresponding point on G has coordinates

   $$(.10, 2^{.10}) \approx (.10, 1.0718).$$

M_2 has coordinates (.20, 1.1493), since

$$\frac{1}{2}(1.10957 + 1.18921) \approx 1.14939.$$

The corresponding point on G has coordinates

$$(.20, 2^{.20}) \approx (.20, 1.1487).$$

Thus, three points of G which are below the chord $\overline{PQ}$ are

$$(.15, 2^{.15}), \quad (.10, 2^{.10}), \quad \text{and} \quad (.20, 2^{.20}).$$

3. Sketch is similar to that of Figure 4-5b.

4. Given

$$P(.05, 4^{.05}) = (.05, 2^{.10})$$

and

$$Q(.25, 4^{.25}) = (.25, 2^{.5}).$$

Point M has coordinates

$$(.15, \frac{2^{.1} + 2^{.5}}{2}) \approx (.15, 1.2430).$$

The corresponding point R on G has coordinates

$$R(.15, 4^{.15}) = (.15, 2^{.3}) \approx (.15, 1.23114).$$

Point M_1 has coordinates (.10, 1.15145), approximately, since

$$\frac{1}{2}(1.07177 + 1.23114) \approx 1.151455.$$

The corresponding point on G has coordinates

$$(.10, 4^{.10}) = (.10, 2^{.2}) \approx (.1, 1.14870).$$

Point M_2 has coordinates (.20, 1.32267), approximately, since

$$\frac{1}{2}(1.23114 + 1.41421) \approx 1.322675.$$

The corresponding point on G has coordinates

$$(.20, 4^{.2}) = (.2, 2^{.4}) \approx (.2, 1.31951).$$

Three points of G which are below the chord $\overline{PQ}$ are

$$(.15,\ 4^{.15}), \qquad (.10,\ 4^{.10}), \qquad (20,\ 4^{.20}).$$

5. From (1) we have

$$\frac{y_1 + y_2}{2} \geq \sqrt{y_1 \cdot y_2} \quad \text{and} \quad \frac{y_3 + y_4}{2} \geq \sqrt{y_3 \cdot y_4}$$

Also,

$$\frac{1}{2}\left(\frac{y_1 + y_2}{2} + \frac{y_3 + y_4}{2}\right) \geq \left[\sqrt{y_1 \cdot y_2}\ \sqrt{y_3 \cdot y_4}\right]^{\frac{1}{2}}$$

or

$$\frac{1}{4}(y_1 + y_2 + y_3 + y_4) \quad \geq \sqrt[4]{y_1 \cdot y_2 \cdot y_3 \cdot y_4}$$

*6. We set $\frac{1}{3}(y_1 + y_2 + y_3) = y_4$ and substitute in the result obtained in Exercise 5, thus,

$$\frac{1}{4}(y_1 + y_2 + y_3) + \frac{1}{12}(y_1 + y_2 + y_3)$$

$$\geq \sqrt[4]{y_1 \cdot y_2 \cdot y_3} \cdot \sqrt[4]{\frac{1}{3}(y_1 + y_2 + y_3)} \quad ,$$

or

$$\frac{1}{3}(y_1 + y_2 + y_3) \geq \sqrt[4]{y_1 \cdot y_2 \cdot y_3} \cdot \sqrt[4]{\frac{1}{3}(y_1 + y_2 + y_3)} \quad .$$

Dividing by the second factor on the right gives

$$\left(\frac{y_1 + y_2 + y_3}{3}\right)^{\frac{3}{4}} \geq (y_1 \cdot y_2 \cdot y_3)^{\frac{1}{4}}$$

and therefore

$$\frac{1}{3}(y_1 + y_2 + y_3) \geq \sqrt[3]{y_1 \cdot y_2 \cdot y_3} \ .$$

4-6 and 4-7. Tangent Lines to Exponential Graphs. Pages 170-179.

In these sections, a tangent line is treated in the spirit of Chapter 3 as the best linear approximation to the graph at a point. Since our development rests upon the use of 2 as base, we begin with the graph of f: $x \longrightarrow 2^x$ and, as in Chapter 3, with the point $(0, f(0)) = (0,1)$.

We show that above an interval $[-b,b]$ of the x-axis, the graph is confined to a wedge formed by two intersecting straight lines. The proof makes use of the concavity of the graph (Section 4-5), the fact that $2^{-x} = \frac{1}{2^x}$, and operations with inequalities which should be familiar. The results for $b = .01$ and $.001$ make it plausible that the best linear approximation exists and that its equation is $y = 1 + kx$ where $k \approx 0.693$. Since k plays an important role in the following development, it is well to emphasize its importance.

The choice of $e = 2^{1/k}$ as a base rests upon the fact that for $|x|$ small,

$$(2^{1/k})^x = 2^{x/k} \approx 1 + k(\tfrac{x}{k}) = 1 + x ,$$

so that $e^x \approx 1 + x$. It follows that for a large positive integer n, $e^{1/n} \approx 1 + \frac{1}{n}$ and $e \approx (1 + \frac{1}{n})^n$. This corresponds to the conventional definition $e = \lim\limits_{n\to\infty} (1 + \frac{1}{n})^n$ which uses a language different from that of this text. We believe that our treatment is an intuitively suggestive one.

Answers to Exercises 4-6a. Page 173.

1. We are to show at which steps in the proof on pages 171-172, it is necessary to assume that

   (a) $mx \neq -1$, (b) $mx \neq 0$, and (c) $mx \neq 1$.

   We are also to show that none of these possibilities can occur.

   (a) In the step

   $$\frac{1}{2^x} > \frac{1}{1 + mx} ,$$

   we must assume that $mx \neq -1$, in order to avoid division by 0. This assumption is justified from the fact that

   $$2^x < 1 + mx \quad \text{for} \quad 0 < x < b.$$

If $mx = -1$, then we would have $2^x < 0$, which we know is false, since $2^x > 0$ for all x. Therefore $mx = -1$ cannot occur.

(b) In the step $1 - m^2x^2 < 1$, we must assume that $mx \neq 0$ in order to avoid the statement $1 < 1$. The fact that mx cannot be 0 follows from the fact that

$$2^x < 1 + mx, \qquad 0 < x < b .$$

If $mx = 0$, then we would have $2^x < 1$. But for $x > 0$, we know that $2^x > 1$. Hence, $mx = 0$ cannot occur.

(c) In the step

$$2^{-x} > \frac{1\cdot(1 - mx)}{(1 + mx)(1 - mx)},$$

we must assume that $mx \neq 1$ in order to avoid division by 0. To show that $mx \neq 1$, we need the given restriction that $0 < x < b < 1$. If $mx = 1$, then $y = 1 + mx = 2$. But $y = 2$ is the ordinate of a point on the graph of

$$g: x \rightarrow 2^x ,$$

and hence

$$2^x = 2$$

at this point. This implies that $x = 1$, which contradicts the given fact that $x < 1$. Therefore, $mx = 1$ cannot occur.

2. If $\overline{m}$ is the slope of L_2 (the line $\overleftrightarrow{PR}$), we are required to prove that G lies above L_2 for $0 < x < b$; that is, that

$$2^x > 1 + \overline{m}x \quad \text{for} \quad 0 < x < b;$$

This is equivalent to the statement that

$$2^{-x} > 1 - \overline{m}x \quad \text{for} \ -b < x < 0.$$

Now,

$$2^{-x} = \frac{1}{2^x} \quad \text{and} \quad 2^x < 1 + \overline{m}x, \ -b < x < 0.$$

(The first statement is a definition; the second statement follows from the fact that G, the graph of $g: x \longrightarrow 2^x$, is concave upward.)

Hence,

$$2^{-x} = \frac{1}{2^x} > \frac{1}{1 + \overline{m}x}$$

Then

$$2^{-x} > \frac{1\cdot(1 - \overline{m}x)}{(1 + \overline{m}x)(1 - \overline{m}x)} = \frac{1 - \overline{m}x}{1 - \overline{m}^2x^2} .$$

Since

$$1 - \overline{m}^2x^2 < 1 \quad (\text{because } \overline{m}x > 0),$$

$$2^{-x} > 1 - \overline{m}x,$$

which is the required conclusion. Therefore, G lies above L_2 for $0 < x < b$.

3. $2^{0.001} \approx 1.0006934$ and $2^{-0.001} \approx 0.9993071$.
The slope of L_1 is

$$\frac{2^{0.001} - 1}{0.001} \approx 0.6934, \text{ and}$$

the slope of L_2 is

$$\frac{1 - 2^{-0.001}}{0.001} \approx 0.6929$$

4. Since for $|x| < 0.001$, the graph of $x \longrightarrow 2^x$ lies in the region between lines L_1 and L_2, the slope k of the tangent to G at $P(0, 1)$ satisfies the inequality

$$0.6929 < k < 0.6934.$$

5. Using $b = 0.01$ we have

$$4^{0.01} = 2^{0.02} \approx 1.01396$$

and

$$4^{-0.01} = 2^{-0.02} \approx 0.98623;$$

hence the slope of L_1 is

$$\frac{2^{0.02} - 1}{0.01} \approx 1.396$$

and the slope of L_2 is

$$\frac{1 - 2^{-0.02}}{0.01} \approx 1.377.$$

Using $b = 0.001$ gives the slope of L_1 as

$$\frac{4^{0.001} - 1}{0.001} \approx \frac{1.001388 - 1}{0.001} = 1.388$$

and the slope of L_2 as

$$\frac{1 - 4^{-0.001}}{0.001} = \frac{1 - 2^{-0.002}}{0.001} \approx \frac{1 - 0.998615}{0.001} = 1.385$$

The slope of the tangent to $x \longrightarrow 4^x$ at the point $P(0, 1)$ is between 1.385 and 1.388. It is 2k.

6. Since $3 \approx 2^{1.59}$ (see Example 4 in Section 4-3)

$$3^{0.1} \approx 2^{.159} = (2^{.15})(2^{.009})$$

$$\approx (1.10957)(1.00626) \approx 1.1165$$

and

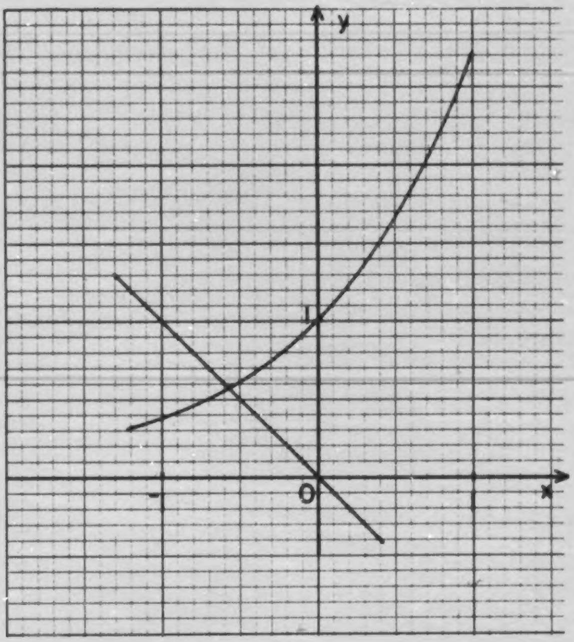
$$3^{-0.1} \approx 2^{-.159} = (2^{-.15})(2^{-.009})$$

$$\approx (0.90125)(0.99378) \approx 0.8956.$$

Thus the slope of the tangent to the graph of $x \longrightarrow 3^x$ at $P(0, 1)$ is between 1.044 and 1.165.

Answers to Exercises 4-6b. Page 178.

1. The graphs of $f\colon x \longrightarrow e^x$ and $g\colon x \longrightarrow -x$ intersect in a unique point. See graph.



Exercise 1

The graphs of f: $x \longrightarrow e^x$
and g: $x \longrightarrow -x$

2. $$f(x) = e^x + x \quad \text{and} \quad f'(x) = e^x + 1$$

Take $$x_1 = -0.55.$$

Since $$f(-.55) \approx 0.5770 - 0.55 = 0.027$$
$$f'(-.55) \approx 1.5770,$$

then $$x_2 \approx -.55 - \frac{0.027}{1.577} \approx -0.567 \text{ or } -0.57.$$

Since
$$e^{-0.57} = (e^{-0.55})(e^{-0.02})$$
$$\approx (0.577)(0.9802) \approx 0.5655,$$
$$f(-0.57) = -.0045 \quad \text{and} \quad f'(-0.57) = 1.5655.$$

Thus $$x_3 \approx -0.57 + \frac{.0045}{1.5655} \approx -0.57 + 0.003 = -0.567$$

The required root is -0.57 correct to two decimals.

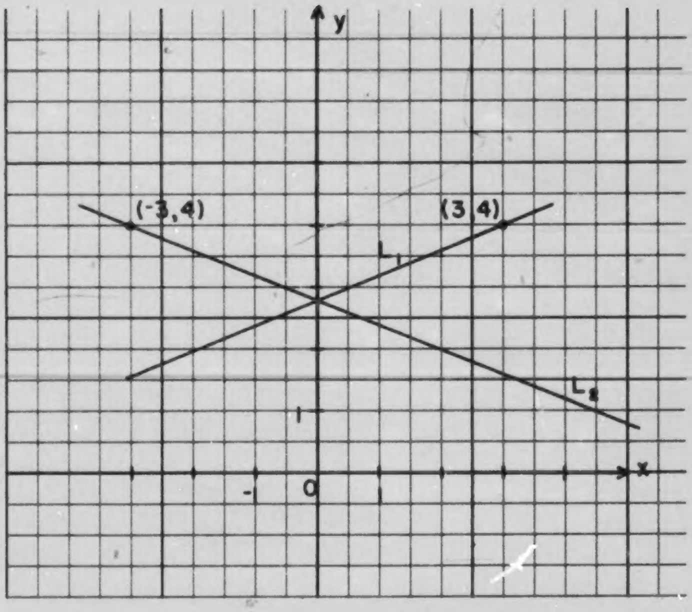
Answers to Exercises 4-7. Pages 179-180.

1. a) $m = e^{-1} \approx 0.7679$
   b) $m = e^{0.5} \approx 1.6487$
   c) $m = e^{0.7} \approx 2.0138$
   d) $m = 1$
   e) $m = e^{1.5} \approx 4.4817$

2. See Figure 4-6b.

3. a) $y = e^{-1}(x + 1) + e^{-1} = e^{-1}(x + 2)$
   b) $y = e^{0.5}(x - .5) + e^{0.5} = e^{0.5}(x + .5)$
   c) $y = e^{0.7}(x - .7) + e^{0.7} = e^{0.7}(x + .3)$
   d) $y = x + 1$
   e) $y = e^{1.5}(x - 1.5) + e^{1.5} = e^{1.5}(x - .5)$

4.



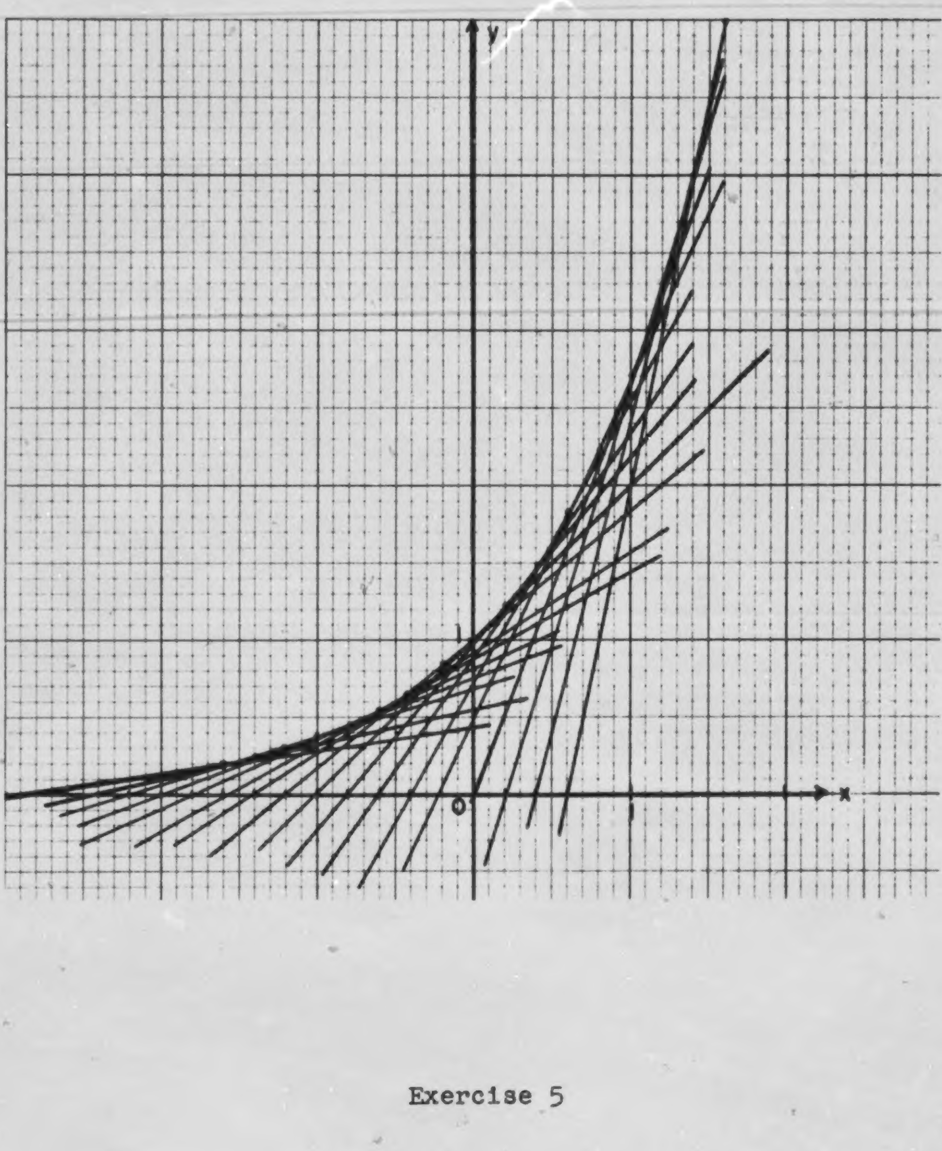
Exercise 4

c) Point (-3,4)

d) The slope of L_2 is $-\frac{2}{5}$.

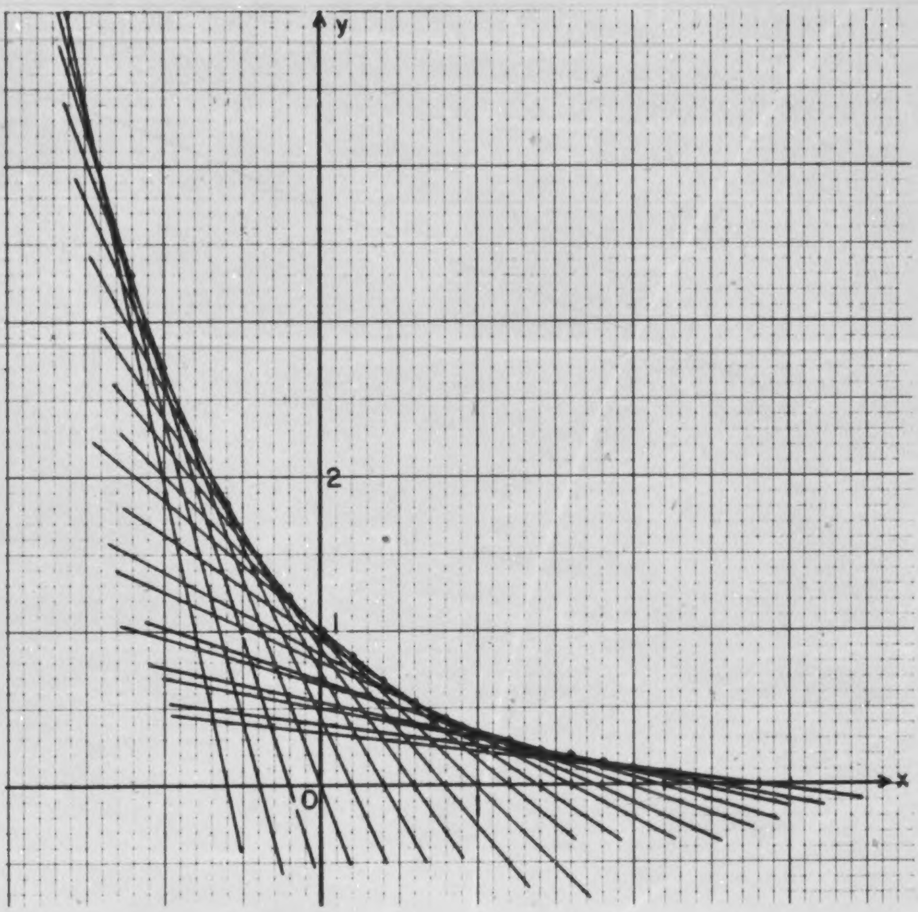
e) Point (-r,s) on L_2 corresponds to point (r,s) on L_1. The slope of L_2 is -m.

5.



Exercise 5

6. a)



Exercise 6

b) Each point has coordinates $(-x, e^{-x})$

c) The slope of each line drawn in 6(a) is the negative the slope of corresponding line drawn in 5(b).

7. a) See Figure 4-6b for the graph of $f: x \to e^x$.

The graph of $g: x \to e^{-x}$ may be obtained by reflecting the graph of f in the y-axis.

b)

	Slope of graph of f	Slope of graph of g
at $x = 0$	1	-1
at $x = +1$	$e \approx 2.72$	$-\frac{1}{e} \approx -0.37$
at $x = -1$	$\frac{1}{e} \approx 0.37$	$-e \approx -2.72$

c) At $x = h$ the slope of the graph of $g : x \to e^{-x}$ is

$$-e^{-x} = -g(x).$$

*8. $g': x \longrightarrow -e^{-x}$

Answers to Exercises 4-8a. Pages 183-184.

1. We use the formula $W(x) = W(0)2^{-x/T}$. Then $\frac{W(x)}{W(0)} = 2^{-x/T}$

so that $\frac{W(7.7)}{W(0)} = 2^{-\frac{7.7}{3.85}} = 2^{-2} = \frac{1}{4}$, and after 7.7 days we would expect 1/4 of the sample to remain. Note that we could have observed that after every 3.85 days we have 1/2 left. Hence, after 2(3.85) days we would expect (1/2)(1/2) = 1/4 left.

Similarly $\frac{W(30.8)}{W(0)} = 2^{-\frac{30.8}{3.85}} = 2^{-8} = \frac{1}{256}$.

2. In this problem we seek $\frac{W(x)}{W(0)} = 2^{-x/T}$ when $T = 26.8$,

$x = 13.4$ and $x = 80.4$, or $x/T = \frac{13.4}{26.8} = \frac{1}{2}$

and

$$x/T = \frac{80.4}{26.8} = 3.$$

Therefore

$$\frac{W(13.4)}{W(0)} = 2^{-1/2} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx .707.$$

$$\frac{W(80.4)}{W(0)} = 2^{-3} = \frac{1}{2^3} = \frac{1}{8} = .125.$$

3. We know $W(x) = W(0)2^{-x/T}$ where $\frac{W(12.2)}{W(0)} = \frac{1}{16}$, and we seek T.

Hence
$$\frac{1}{16} = 2^{-12.2/T} \quad \text{or} \quad 2^{-4} = 2^{-12.2/T} .$$

Hence,
$$-4 = \frac{-12.2}{T} \quad \text{and} \quad T = \frac{12.2}{4} = 3.05.$$

The half-life is 3.05 minutes (approximately).

4. Once again we use
$$W(x) = W(0)2^{-x/T}.$$

We are given that when
$$x = 1, \ \frac{W(1)}{W(0)} = \frac{49}{50} .$$

Therefore,
$$2^{-1/T} = \frac{49}{50} = 0.98.$$

From Table 4-2 we read
$$2^{-.03} \approx 0.98;$$

hence
$$2^{-1/T} \approx 2^{-.03} \quad \text{and} \quad -\frac{1}{T} \approx -.03$$

so that
$$T \approx \frac{100}{3} .$$

Thus $W(x) = 2(2^{\frac{-3x}{100}})$ gives the number of milligrams after x years.

An alternative solution can be based on the fact that

$2^x \approx 1 + kx$ when $|x|$ is near 0:

$2^{-1/T} \approx 1 + k(-\frac{1}{T}) = 1 - \frac{k}{T} \approx 0.98$; hence $T \approx \frac{k}{.02} \approx 34.6$

5.
$$\frac{W(x)}{W(0)} = 2^{-x/T}$$

$$\frac{W(3.36 \cdot 10^4)}{W(0)} = \frac{3}{4} = 2^{\frac{-3.36 \cdot 10^4}{T}}$$

Therefore,

$$\frac{4}{3} \approx 1.333 \approx 2^{0.414} = 2^{\frac{3.36\cdot 10^4}{T}}$$

Hence

$$.414 = \frac{3.36\cdot 10^4}{T}$$

and

$$T = \frac{3.36}{.414}\cdot 10^4 \approx 8.15\cdot 10^4$$

6. $$\frac{W(x)}{W(0)} = 2^{-x/T}$$

$$\frac{W(3{,}000)}{W(0)} = \frac{.277m}{m} = .277 \approx 2^{\frac{-3{,}000}{T}}$$

Hence

$$2^{\frac{3{,}000}{T}} \approx \frac{1}{.277} \approx 3.614 = 2\cdot 1.807$$

$$\approx 2^{1}\cdot 2^{.85} = 2^{1.85},$$

and

$$\frac{3{,}000}{T} \approx 1.85, \quad T = \frac{3{,}000}{1.85} \approx 1{,}620 \text{ years.}$$

Hence

$$\frac{W(x)}{W(0)} = 2^{-x/1{,}620}; \quad W(0) = 2 \text{ milligrams}$$

$$W(810) = 2\cdot 2^{-810/1{,}620} = 2\cdot 2^{-.5}$$

$$= 2^{.5} = \sqrt{2} \approx 1.4.$$

Thus, 1.4 milligrams (approximately) will remain after 81 decades.

Answers to Exercises 4-8b. Page 186.

1. We use the formula $A = Pe^{rt}$. Then

$$A = 1000\cdot e^{(.03)18} = 1000\ e^{.54}.$$

Using Table 4-6 and interpolating we write

$$e^{.54} \approx 1.7164.$$

Thus

$$A = 1000\ e^{.54} \approx 1000(1.7164) \approx 1716;$$

the amount is \$1,716 (approximately).

2. a) $A = Pe^{rt}$. Since the amount $A = 2P$, and $r = .03$, we have $2P = P\cdot e^{.03t}$ and $e^{.03t} = 2$. Using $2 \approx e^{.693}$, we have $e^{.03t} \approx e^{.693}$ so that $t \approx 23.1$.

   The time required is 23 years (approximately).

   b) Using $r = 0.06$, we have

   $$2P = P\cdot e^{.06t} \quad \text{and} \quad e^{.06t} = 2 \approx e^{.693}$$

   Thus $t \approx 11.5$; the required time is 12 years. (approximately).

   c) Using $r = n/100$, we have

   $$2P = Pe^{nt/100} \quad \text{and} \quad e^{nt/100} \approx e^{.693};$$

   hence

   $$t \approx \frac{69.3}{n}.$$

Answers to Exercises 4-8c. Pages 188-189.

1. Since $P = 180$ and $P_0 = 760$, $180 = 760e^{-0.11445h}$, and $e^{-0.1144h} = \frac{180}{760} \approx 0.237$. From Table 4-6 we see that $0.237 \approx e^{-1.45}$; hence $-0.1144h \approx -1.45$, and $h \approx 12.7$. Thus the height is about 12.7 kilometers.

2. $Q = \frac{Q_0}{2} = Q_0e^{-0.12n}$ so that $\frac{1}{2} = e^{-0.12n}$. Since $0.5 \approx e^{-0.69}$, $-0.12n \approx -0.69$ and $n \approx 5\frac{3}{4}$. The required time is about $5\frac{3}{4}$ days.

3. $W(x) = T(x) - B$; hence $W(0) = 100 - 20 = 80$,
   $W(5) = 90 - 20 = 70$, and $W(x) = 30 - 20 = 10$.

Using the formula $W(x) = W(0)e^{-cx}$ we have $70 = 80\ e^{-5c}$. Since $\frac{70}{80} = 0.875 \approx e^{-0.13}$, $-5c \approx -0.13$ and $c \approx 0.026$, so that $W(x) \approx 80e^{-0.26x}$.

But $W(x) = 10 = 80e^{-0.26x} \Longleftrightarrow e^{-0.26x} \approx \frac{1}{8} \approx e^{-2.08}$.

Hence $-0.26x \approx -2.08$ and $x \approx 7.7$. The time required is about 7.7 minutes.

4. $I(5) = I_0e^{-5k} = \frac{2}{3}I_0$, so that $e^{-5k} = \frac{2}{3}$.

$I(10) = I_0e^{-10k} = I_0(e^{-5k})^2 = 4/9\ I_0$.

Hence, 10 feet below the surface the intensity is $\frac{4}{9}I_0$.

[As an alternative method, we note that

$$\frac{2}{3} = \frac{I(5)}{I(0)} = \frac{I(5 + 5)}{I(0 + 5)} = \frac{I(10)}{I(5)}.$$

It follows that

$$I(5) = \frac{2}{3}I_0 \quad \text{and} \quad I(10) = \frac{2}{3}I(5) = \left(\frac{2}{3}\right)^2 I_0 .$$]

If $I(x) = \frac{1}{2}I_0$, $\frac{1}{2} = e^{-kx}$. From above, $e^{-5k} = \frac{2}{3} \approx e^{-0.40}$, so that $k \approx 0.08$. Thus $e^{-0.08x} \approx 0.5 \approx e^{-0.70}$ and $x \approx 8.75$; hence at depth of about $8\frac{3}{4}$ feet the intensity is $\frac{1}{2}I_0$.

Answers to Exercises 4-9. Pages 194-196.

1. a) $x \longrightarrow \frac{x + 5}{4}$

   b) $x \longrightarrow \frac{3}{x - 8}$

   c) $x \longrightarrow \sqrt[3]{x + 2}$

2. a) $x = \frac{y + 5}{4}$ b) $x = \frac{3}{y - 8}$ c) $x = \sqrt[3]{y + 2}$

3. Suppose the digits are x and y, and we pick x. Then we define:

$$f_1: x \longrightarrow 5x$$
$$f_2: x \longrightarrow x + 7$$
$$f_3: x \longrightarrow 2x$$
$$f_4: x \longrightarrow x + y$$
$$f_5: x \longrightarrow x - 14$$

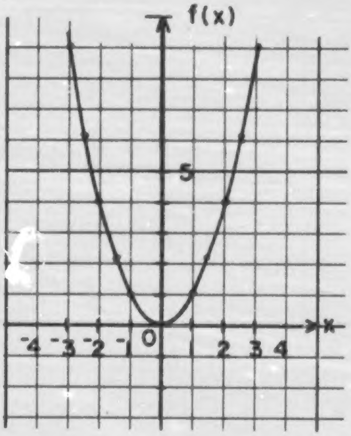
$$f_5f_4f_3f_2f_1: x \longrightarrow 2(5x + 7) + y - 14 = 10x + y,$$

a number with tens digit x and units digit y.

4. A function which has an inverse.

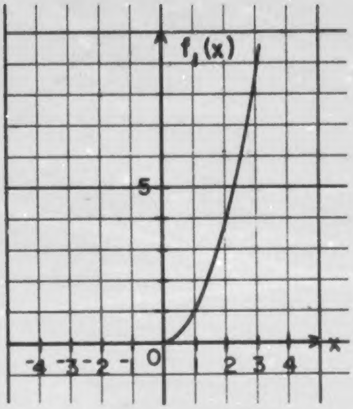
5. If $x_1 \neq x_2$, then either $x_1 < x_2$, in which case $f(x_1) > f(x_2)$, or $x_1 > x_2$, in which case $f(x_1) < f(x_2)$. In either case, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$; hence f is one-to-one and has an inverse by Corollary 4-2-1.

6. a) $f(1) = 1 = f(-1)$ suffices to show that f is not one-to-one and therefore does not have an inverse.

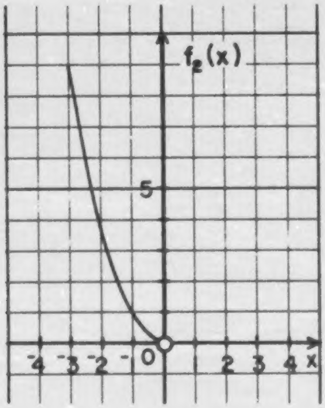


$f: x \longrightarrow x^2$

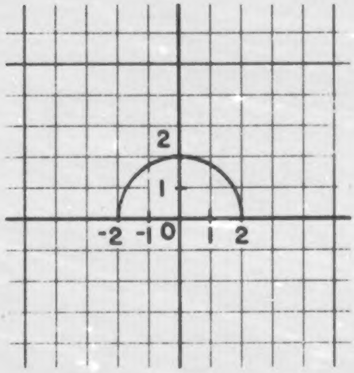
b) $f_1^{-1}: x \longrightarrow \sqrt{x}$



c) $f_2^{-1}: x \longrightarrow \sqrt{-x}$



7. a) $f(1) = \sqrt{3} = f(-1)$ suffices to show that f is not one-to-one and therefore does not have an inverse.



$f: x \longrightarrow \sqrt{4 - x^2}$

b) For example, $f_1\colon x \rightarrow \sqrt{4 - x^2}$, $0 \le x \le 2$,
and
$f_2\colon x \rightarrow \sqrt{4 - x^2}$, $-2 \le x \le 0$.

8. $f_1\colon x \longrightarrow x^2 - 4x$, $x \geq 2$, and $f_2\colon x \longrightarrow x^2 - 4x$, $x < 2$.

9. All $x \longrightarrow x^3 - 3x$, with domains $\{x\colon x < -1\}$, $\{x\colon -1 \le x \le 1\}$, and $\{x\colon x > 1\}$.

10. If f has an inverse, no line parallel to the x-axis may meet the graph of f in more than one point. This implies, for a cubic, that it cannot have a maximum and a minimum. Hence the slope function f' has only one real zero (graph of f has a point of inflection with horizontal tangent) or none.

4-10. Logarithms. Pages 196-203.

If f is defined by $f\colon x \longrightarrow 2^x$ we have, in particular, $f\colon 3 \longrightarrow 2^3$ or $f\colon 3 \longrightarrow 8$. Thus, under f, 3 is associated with 8. Under f^{-1} then 8 is associated with 3, or $f^{-1}\colon 8 \longrightarrow 3$. In general, $f^{-1}\colon 2^x \longrightarrow x$. This form is neither convenient nor in conformity with our way of writing a function. We prefer to write $f^{-1}\colon x \longrightarrow y$ and represent y in some manner in terms of x. We get around this by using a new symbol, $\log_2$, for f^{-1} so that we may now write $f^{-1}\colon x \longrightarrow \log_2 x$. Thus $\log_2 8 = 3$ is simply another way of writing $2^3 = 8$. More generally, for any positive real numbers, a, b and c,

$$\log_a c = b \Longleftrightarrow a^b = c.$$

These two equivalent forms express the same relation among a, b, and c. For example, to find $\log_4 8$ we may set $\log_4 8 = x$, write $4^x = 8 \Longleftrightarrow 2^{2x} = 2^3$, and obtain $x = \frac{3}{2}$.

On the other hand if we use the notation exp and log for the functions f and f^{-1}, respectively, then $\exp_a: x \longrightarrow a^x$ and $\log_a: x \longrightarrow \log_a x$. Furthermore, since exp and log are inverse functions

$$\exp_a(\log_a x) = x \quad \text{and} \quad \log_a(\exp_a x) = x.$$

That is,

$$a^{\log_a x} = x \qquad \text{and} \quad \log_a a^x = x.$$

Although the exp notation is not used in the text, it may be introduced at the discretion of the teacher.

Answers to Exercises 4-10. Pages 204-205.

1. $\log_e 10 = \log_e(5\cdot 2) = \log_e 5 + \log_e 2 \approx 1.609 + 0.693 = 2.302$
2. $\log_e 5/4 \approx \log_e 1.2214 + \log_e 1.0202 + \log_e 1.003$

   $\approx 0.20 + 0.02 + 0.003 = 0.223$
3. $\log_e 4 = \log_e 2^2 = 2 \log_e 2 \approx 1.386$
4. a) $\log_e 5 = \log_e (4)(5/4) = \log_e 4 + \log_e 5/4 \approx 1.386 + 0.223 =$ 1.609

   b) the results are the same.
5. $\log_e 3 = \log_e 2.7183 + \log_e 1.0513 + \log_e 1.0408 + \log_e 1.008$

   $\approx 1.0 + 0.05 + 0.04 + 0.008 = 1.098$
6. $\log_e 0.25 = \log_e 2^{-2} = -2 \log_e 2 = -1.386$

   $\log_e 0.5 = \log_e 2^{-1} = -\log_e 2 = -0.693$

   $\log_e 2/3 = \log_e 2 - \log_e 3 = 0.693 - 1.098 = -0.405$

   $\log_e 5/3 = \log_e 5 - \log_e 3 = 1.609 - 1.098 = 0.511$

   $\log_e 2.5 = \log_e 5 - \log_e 2 = 1.609 - 0.693 = 0.916$

   $\log_e 6 = \log_e 3 + \log_e 2 = 1.098 + 0.693 = 1.791$

$\log_e 8 = \log_e 2^3 = 3 \log_e 2 = 2.079$

$\log_e 9 = \log_e 3^2 = 2 \log_e 3 = 2.196$

9. $32 = 4^x \Longleftrightarrow x = 5/2$

10. $a \cdot a^m = a^{1+m} = (a^2)^m = a^{2m} \Longleftrightarrow m = 1$

11. $\log_a(x \cdot \frac{1}{x}) = \log_a 1 = 0$. For any real number $x > 0$,

$\log_a(x \cdot \frac{1}{x}) = \log_a x + \log_a(\frac{1}{x})$; since $\log_a x + \log_a(\frac{1}{x}) = 0$,

$\log_a(\frac{1}{x}) = -\log_a x$ for $x > 0$.

12. For any real numbers $x_1 > 0$, $x_2 > 0$,

$$\log_a(\frac{x_1}{x_2}) = \log_a(x_1)(\frac{1}{x_2}) = \log_a x_1 + \log_a(\frac{1}{x_2})$$

$$= \log_a x_1 - \log_a x_2, \text{ from Exercise 11.}$$

13. If $f: x \longrightarrow a^x$, $f(1) = a$, $f^{-1}(a) = 1$. In other words $\log_a a = 1$.

14. a) $10^y = 35$

b) $2^x = 25$

c) $e^b = d$

d) $\log_{10} 5^2 = x$, so that $10^x = 25$.

e) $\log_e(6)(7) = x$, so that $e^x = 42$.

f) $\log_e 25^{1/2} - \log_e 2 = \log_e 5/2 = x$; so that $e^x = 2.5$.

15. $\log_{10} 5 = \log_{10} 10 - \log_{10} 2 \approx 1 - 0.3010 = 0.6990$

$\log_{10}(1/2) = \log_{10} 1 - \log_{10} 2 \approx 0 - 0.3010 = -0.3010$

$\log_{10}(\frac{25}{4}) = \log_{10} 5^2 - \log_{10} 2^2 = 2 \log_{10} 5 - 2 \log_{10} 2 \approx 0.7960$

$\log_{10}(\frac{128}{5}) = 7 \log_{10} 2 - \log_{10} 5 \approx 2.1070 - 0.6990 = 1.4080$

16. a) $\log_{125} 5 = \frac{1}{3}$

b) $\log_{10}0.01 = -2$

c) $\log_{27}81 = \frac{4}{3}$

d) $\log_{0.04}0.008 = \frac{3}{2}$

e) $\log_{16}2 = \frac{1}{4}$

17. $\log_6(x + 9)(x) = \log_6 36$, hence $x^2 + 9x = 36$.

$x^2 + 9x - 36 = 0 \Longleftrightarrow x = 3$ or $x = -12$.

The only root of the given equation is 3.

4-11. Special Bases for Logarithms. Pages 205-206.

Some attention might well be devoted to the function

$$x \longrightarrow \log_2 x.$$

The graph may be readily obtained from that of $x \longrightarrow 2^x$ (Figure 4-3c), since these graphs are symmetric with respect to the line $y = x$. If the graph of $x \rightarrow \ln x$ is then sketched (or traced) on the same coordinate system, an examination of the graphs for various values of x clearly shows that

$$\ln x = \frac{\log_2 x}{\log_2 e} \quad \text{or} \quad \frac{\ln x}{1} = \frac{\log_2 x}{\log_2 e}.$$

It is desirable that the student gain some measure of confidence in dealing with logarithmic functions involving different bases.

Answers to Exercises 4-11. Pages 207-209.

1. a) $2^p = 26 \Longleftrightarrow p = \log_2 26$.

b) $\log_e x = 5 \Longleftrightarrow e^5 = x \approx 148.41$

c) $\log_3(3^{1/4}) = 1/4$

d) $\log_2 (8 \times 16) = \log_2 2^7 = 7$

2. a) $\ln 4 = 2 \ln 2 = 2r$

b) $\ln 6 = \ln 2 + \ln 3 = r + s$

c) $\ln \frac{1}{8} = -3 \ln 2 = -3r$

d) $\ln 10 = r + t$

e) $\ln 2.5 = \ln 5 - \ln 2 = t - r$

f) $\ln \frac{2}{9} = \ln 2 - 2 \ln 3 = r - 2s$

g) $\ln \frac{5}{9}\sqrt{3} = \ln 5 - 2 \ln 3 + \frac{1}{2} \ln 3 = t - \frac{3}{2}s$

h) $\ln 8 \sqrt[3]{100} = 3 \ln 2 + \frac{2}{3} (\ln 2 + \ln 5)$

$= 3r + \frac{2}{3} (r + t) = \frac{1}{3}(11r + 2t)$

3. a) $\log_{10}1000 = 3$

b) $\log_{0.01}0.001 = x \Longleftrightarrow (0.01)^x = (10^{-2})^x = 10^{-3} \Longleftrightarrow x = 1.5$

c) $\log_3(\frac{1}{81}) = \log_3(3^{-4}) = -4$

d) $\log_4(32) = x \Longleftrightarrow 4^x = 2^5 \Longleftrightarrow x = 2.5$

e) $\log_{10}(0.0001) = -4$ since $10^{-4} = 0.0001$

f) $\log_{0.5}16 = x \Longleftrightarrow (\frac{1}{2})^x = 2^{-x} = 2^4 \Longleftrightarrow x = -4$

g) $\ln e^3 = 3$

h) $\ln \sqrt{e} = \frac{1}{2}$ since $e^{\frac{1}{2}} = \sqrt{e}$

i) $\log_{81}27 = x \Longleftrightarrow (3^4)^x = 3^3 \Longleftrightarrow x = \frac{3}{4}$

j) $\log_2 \sqrt{32} = x \Longleftrightarrow 2^x = (2^5)^{\frac{1}{2}} \Longleftrightarrow x = \frac{5}{2}$

4. a) $\log_{10}3 = \frac{\ln 3}{\ln 10} \approx \frac{1.0986}{2.3026} \approx 0.477$

b) $\log_{10}e = \frac{\ln e}{\ln 10} \approx \frac{1}{2.3026} \approx 0.434$

c) $\log_3 10 = \frac{\ln 10}{\ln 3} \approx \frac{2.3026}{1.0986} \approx 2.096$

d) $\ln 100 = \ln 10^2 = 2 \ln 10 \approx 4.605$

e) $\ln 30 = \ln 3 + \ln 10 \approx 3.401$

f) $\ln 300 = \ln 3 + 2 \ln 10 \approx 5.704$

g) $\ln 0.3 = \ln 3 + \ln 10^{-1} \approx 1.0986 + (-2.3026) \approx -1.204$

h) $\ln 0.003 = \ln 3 + \ln 10^{-3} \approx 1.0986 + (-3)(2.3026) \approx -5.809$

5. a) $5 + 5 = x \Longleftrightarrow x = 10$

b) $\log_{10}\frac{x^2-1}{(x-1)^2} = \log_{10}\frac{x+1}{x-1} = \log_{10}3 \Longleftrightarrow x + 1 = 3(x-1)$
and $x = 2$

c) $(\log_x 5)(\log_7 7) = \log_7 5 \Longleftrightarrow x = 7$

6. a) 1

b) $\frac{1}{e}$

c) e

d) $e^3 = x-2 \Longleftrightarrow x = 2+e^3$

e) e^{-3}

f) $e^{-2} = 2x - 1 \Longleftrightarrow x = \frac{e^{-2}+1}{2} = \frac{1+e^2}{2e^2}$

7. $e^{\ln 10} = 10$. Hence, $\left(e^{\ln 10}\right)\log_{10}e = 10^{\log_{10}e} = e$.

$$\left(e^{\ln 10}\right)\log_{10}e = e \Longrightarrow e^{\ln 10 \log_{10}e} = e^1$$

$$\Longrightarrow \ln 10 \log_{10}e = 1.$$

8. a) 1

b) For all real $x > 0$.

c) For all real $x > 0$ where $x \neq 1$.

d) $\log_x 2^x = 2 \Longleftrightarrow x^2 = 2^x \Longleftrightarrow x = 2^{x/2}$

Roots are 2, 4, obtained by trial or by graphing $y = x$ and $y = 2^{x/2}$.

9. If $f: x \rightarrow 1^x$, $f(c) = f(d)$ where $c \neq d$, and f has no inverse.

10. $(\ln x)^2 = \ln x^2 \Longleftrightarrow (\ln x)^2 - 2 \ln x = \ln x(\ln x - 2) = 0$

$\ln x = 0 \Longleftrightarrow x = 1$; $\ln x - 2 = 0 \Longleftrightarrow x = e^2$.

Solution set $= \{1, e^2\}$

11. $2N = N(1.04)^x \Longleftrightarrow 2 = (1.04)^x$

$$x = \frac{\log 2}{\log 1.04} \approx \frac{0.30103}{0.01703} \approx 17.7.$$

Time required is 18 years, approximately.

$$2N = N(1.0075)^{4x} \Longleftrightarrow x = \frac{\log 2}{4 \log 1.0075}$$

$$x \approx \frac{0.30103}{4(0.00325)} \approx 23.16$$

Time required is about 23 years.

Note: Although a table of common logarithms was used to obtain the solution, a table of natural logarithms would have served as well.

12. $2N = N(1 + \frac{x}{400})^{4(10)}$ or $2 = (1 + 0.0025x)^{40}$.

$$\log(1 + 0.0025x) = \frac{\log 2}{40} \approx \frac{0.30103}{40} \approx 0.007526 ,$$

hence $1 + 0.0025x \approx 1.0175$ and $x \approx 7$.

Rate is 7%, approximately.

13. $N(x) = N_0 \cdot a^x$ where x is time in hours.

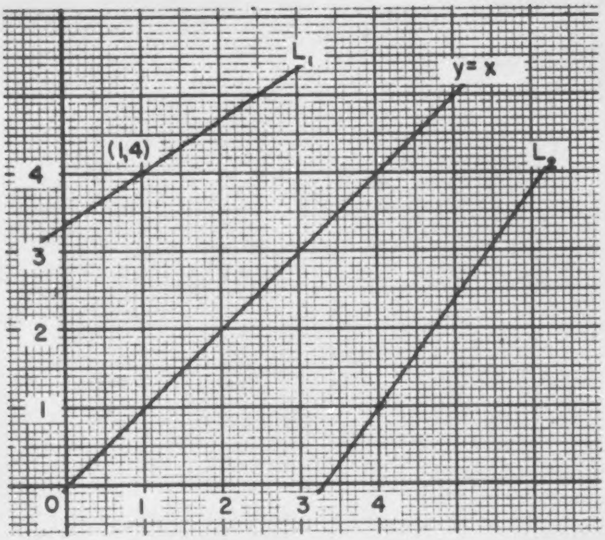
$6 = 1 \cdot a^{3/2}$, so that $\ln a = \frac{2}{3} \ln 6$.

$50 = a^x$ (where $N(x) = 50$ ten thousand) and

$$x = \frac{\ln 50}{\ln a} = \frac{3 \ln 50}{2 \ln 6} \approx \frac{3(3.91202)}{2(1.79176)}$$

$$\approx 3.28 \text{ (hours)}$$

14. a) b)

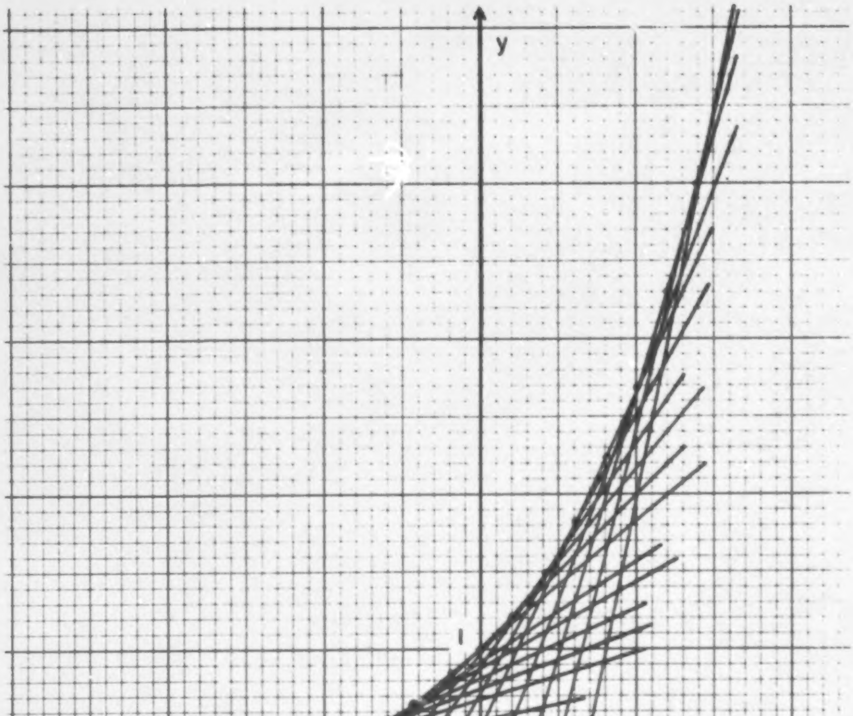


c) (4,1)

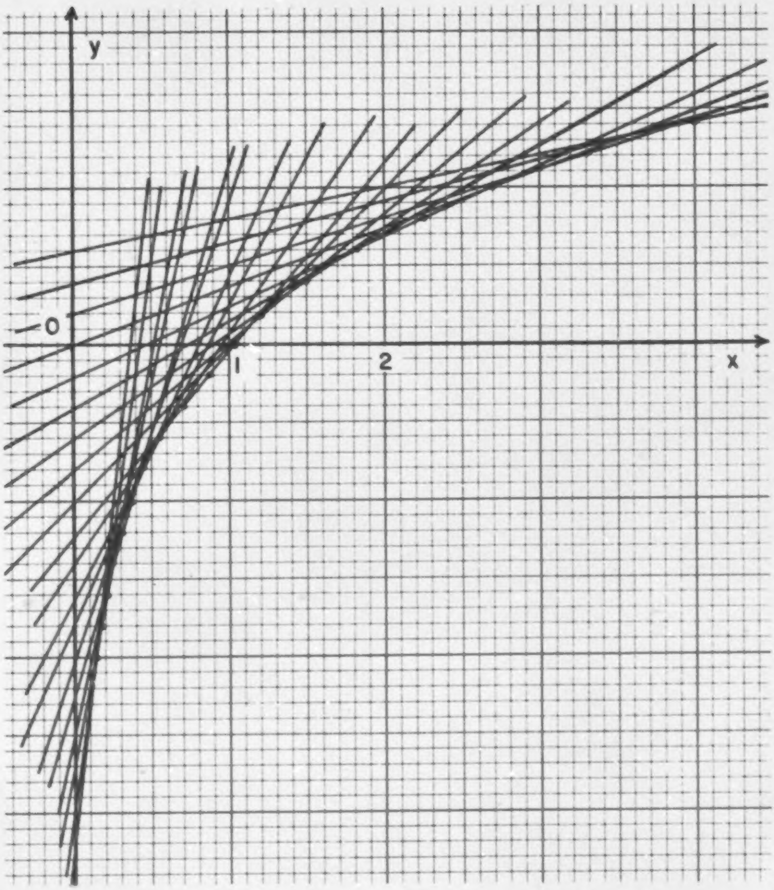
d) 3/2

e) (s,r); $\frac{1}{m}$

15.



16.



17. Slope of graph of $x \longrightarrow e^x$ at $(0,1)$ is 1. Graph of $x \longrightarrow \ln x$ has no slope at $x = 0$. Its slope at $(1,0)$ is 1.

4-12. Computation of e^x and $\ln x$. Pages 210-216.

This section introduces the important idea of approximating the value $f(x)$ of a non-polynomial function f, by substituting values of x in an appropriate polynomial whose values effectively replace $f(x)$ for the purpose of the calculation. In particular, we replace e^x by one of the set of polynomials

$$1 + x$$

$$1 + x + \frac{x^2}{2!}$$

$$\cdot \cdot \cdot \cdot \cdot \cdot \cdot$$

$$1 + x + \frac{x^2}{2!} + \ \ldots + \frac{x^n}{n!}$$

The advantage of this procedure is that the values of these polynomials are easy to find. The use of graphs and a few sample calculations demonstrate the effectiveness of the procedure.

There remain two questions: (1) How are the polynomials discovered? (2) How accurate are the approximations? These questions are not answered in Section 4-12, but are discussed in the Appendix (Section 4-16).

For a further discussion of the computation of $\ln x$ see the Appendix (Section 4-18).

Answers to Exercises 4-12. Page 216.

1. If $x = 0.1$, $e^{0.1} \approx 1 + 0.1 + \frac{(.1)^2}{2} + \frac{(.1)^3}{3!}$

$$\approx 1.1052$$

If $x = 0.5$, $e^{0.5} \approx 1 + 0.5 + \frac{(.5)^2}{2} + \frac{(.5)^3}{3!}$

$$\approx 1.65$$

If $x = -.1$, $e^{-.1} \approx 1 - 0.1 + \frac{(-.1)^2}{2} + \frac{(-.1)^3}{3!}$

$$\approx 0.9048$$

If $x = -.2$, $e^{-.2} \approx 1 - .2 + \frac{(-.2)^2}{2} + \frac{(-.2)^3}{3!}$

≈ 0.8187

2. $1 = 1.0$

$1 = 1.0$

$\frac{1}{2!} = 0.5$

$\frac{1}{3!} \approx 0.16666667$

$\frac{1}{4!} \approx 0.04166667$

$\frac{1}{5!} \approx 0.00833333$

$\frac{1}{6!} \approx 0.00138889$

$\frac{1}{7!} \approx 0.00019841$

$\frac{1}{8!} \approx 0.00002480$

$\frac{1}{9!} \approx 0.00000276$

$\frac{1}{10!} \approx 0.00000028$

Sum $= 2.71828181$

3. The line $y = x-1$ is symmetric to $y = x+1$ with respect to the line $y = x$.

4. $\ln 1.2 \approx (1.2 - 1) - \frac{(1.2 - 1)^2}{2} + \frac{(1.2 - 1)^3}{3}$

$\approx 0.2 - 0.02 + \frac{0.008}{3} \approx 0.18$

From the graph $\ln 1.2 \approx 0.18$. Similarly $\ln 1.3 \approx 0.26$.

5. $\ln 1.21 = \ln(1.1)^2 = 2 \ln 1.1 \approx 0.19062$

Answers to Miscellaneous Exercises. Pages 220-224.

1. Given: $N = Ae^n$

a) If $n = 0$, $N = A$.

$2A = Ae^n \Longrightarrow 2 = e^n$ and $n = \ln 2 = k$

b) Ratio $= e$, hence increase $= e-1$ and per cent increase $= 100(e-1) \approx 172$ per cent.

2. $f: x \longrightarrow ca^x$

$f(0) = c = 2;$ $f(1.5) = 2\, a^{3/2} = 54 \Longleftrightarrow a = 9$

3. $f:x \rightarrow a^x$

   $f(2) = a^2 = \frac{1}{4} \Longleftrightarrow a = \frac{1}{2}$ and

   $f(5) = (\frac{1}{2})^5 = \frac{1}{32}$

4. $V_1 = \sqrt[n]{\frac{V_2{}^n p_1}{p_2}} = V_2 \left(\frac{p_1}{p_2}\right)^{\frac{1}{n}}$

5. $r^n = 1 + \frac{s}{a}(r-1) \Longleftrightarrow n = \frac{\log\left[1 + \frac{s}{a}(r-1)\right]}{\log r}$

6. a) $\ln x^3 = 1 \Longleftrightarrow x = e^{\frac{1}{3}}$

   b) $\ln x^2 - 2 \ln x^{\frac{1}{2}} = \ln \frac{x^2}{x} = \ln x = 1 \Longleftrightarrow x = e$

7. If $a^{0.3} = x$, $0.3 \log_x a = 1$ and $\log_x a = \frac{10}{3}$.

8. a) $\ln (\frac{3}{5})(100)(\frac{1}{12}) = \ln 5$

   b) $\ln \frac{x^2 y^{2/3}}{y^{1/4} x^{1/2}} = \ln(x^{3/2} y^{5/12})$

9. The graphs of $f:x \rightarrow -e^x$ and $g:x \rightarrow e^{-x}$ are symmetric with respect to the origin.

10. a) $f(3^2) = f(9) = 2^9$

    b) $g(2^2) = g(4) = 3^4$

11. a) $f(x) + g(x) = 2(2^x) = 2^{x+1}$

    b) $f(x) \cdot g(x) = 2^{2x} - 2^{-2x}$

    c) $(2^x + 2^{-x})^2 - (2^x - 2^{-x})^2 = 4(2^x)(2^{-x}) = 4$

12. $\log_2(6x + 5)(x) = 2 = \log_2 4 \Longleftrightarrow 6x^2 + 5x = 4$

    The only solution is $\frac{1}{2}$ since $x > 0$.

13. $2^{2x+2} - 9(2^x) + 2 = (2^{2x})(2^2) - 9(2^x) + 2 = 0.$

$$= 4(2^x)^2 - 9(2^x) + 2 = 0$$

Thus, if $y = 2^x$, $4y^2 - 9y + 2 = (4y - 1)(y - 2) = 0$,

and $y = \frac{1}{4}$ or $y = 2$.

The required solutions are -2, and 1.

14. $2^{2x+2} + 2^{x+2} - 3 = 4(2^x)^2 + 4(2^x) - 3 = 0$

If $y = 2^x$, $4y^2 + 4y - 3 = (2y - 1)(2y+3) = 0$ and $y = \frac{1}{2}$ or $y = -\frac{3}{2}$.

The only solution is -1; since $y = 2^x$ we exclude $y = -\frac{3}{2}$.

15. $\ln \frac{x-4}{x+1} = \ln 6 \Longleftrightarrow \frac{x - 4}{x + 1} = 6 \Longleftrightarrow x = -2$

But the domain of $f: x \longrightarrow \ln x$ is the set of positive real numbers, hence $\ln (x-4)$ and $\ln (x+1)$ are undefined for $x = -2$.

16. $\log_b c = \frac{\log_a c}{\log_a b}$ and $\log_c d = \frac{\log_a d}{\log_a c}$

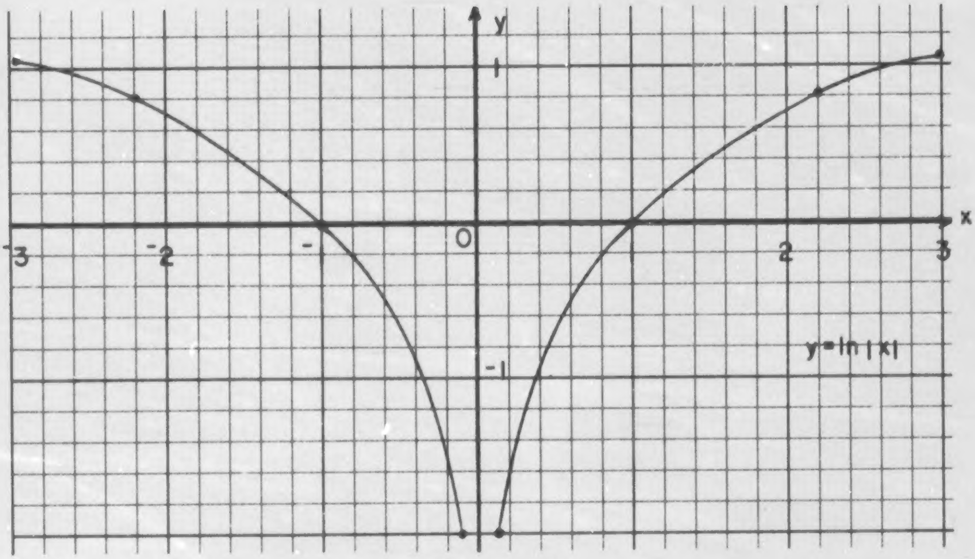
hence

$$(\log_a b)(\log_b c)(\log_c d) = (\log_a b)\left(\frac{\log_a c}{\log_a b}\right)\left(\frac{\log_a d}{\log_a c}\right) = \log_a d .$$

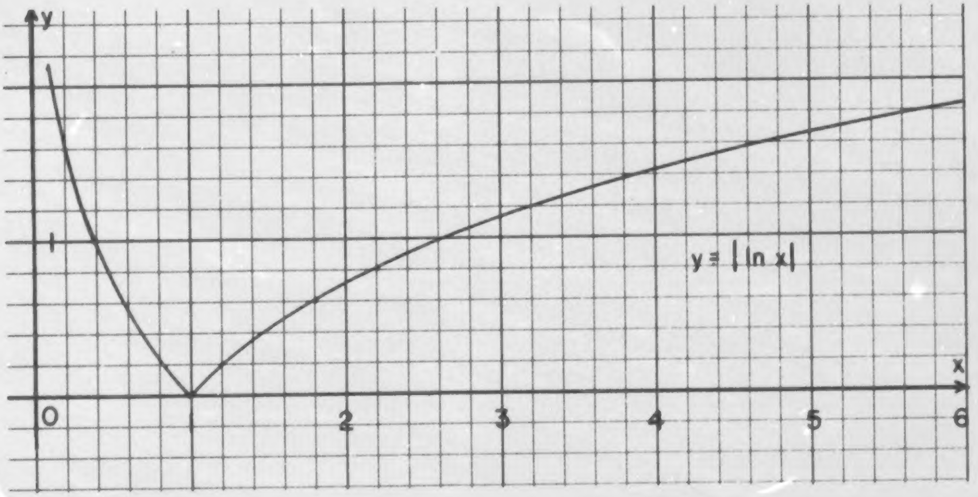
17. $\ln(\frac{1-x}{1+x}) = 1 = \ln e \Longleftrightarrow \frac{1-x}{1+x} = e$ and $x = \frac{1-e}{1+e}$

18. a) $y = \ln |x|$ b) $y = |\ln x|$ c) $y = \ln e^x$

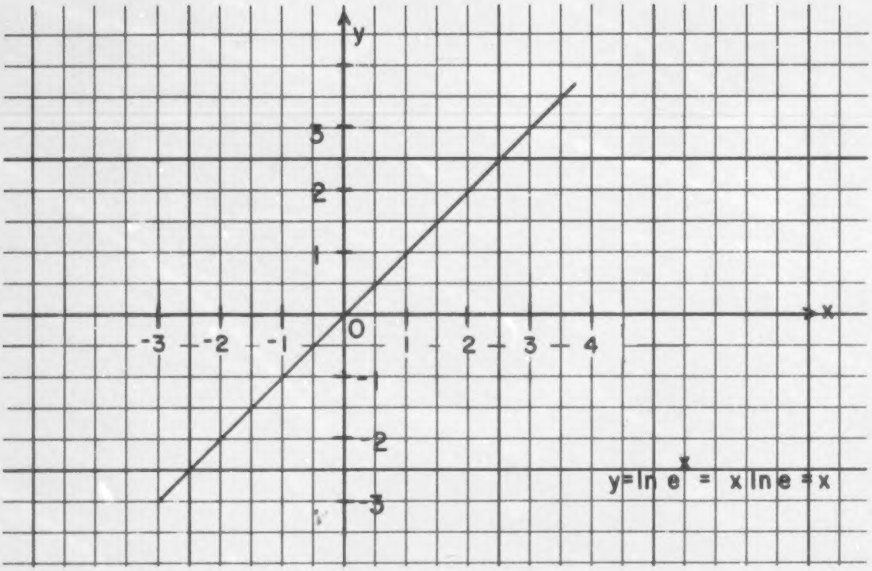
18. a)



b)



c)



19. $A = 100\ e^{5(0.05)} = 100\ e^{0.25} \approx 100\ (1.2840) = 128.40.$
Amount = \$128, approximately

20. $100 = P\ e^{5(0.05)} = P\ e^{0.25} \Longleftrightarrow P = 100\ e^{-0.25}$
$P \approx 100(0.7788) = 77.88$
Answer = \$78, approximately

21. $200 = 100(1 + \frac{x}{100})^{10} \Longleftrightarrow 2 = (1 + 0.01x)^{10}.$

$2^{0.1} = 1 + 0.01\ x \Longleftrightarrow x = 100(2^{0.1}-1) \approx 100(1.0718 - 1)$
Rate is 7.18% or 7.2%, approximately.

22. $N(x) = N_0\ e^{kx}$, for $N(x)$ in thousands.

$25 = N_0$, where 1950 is time at $x = 0$.

$30 = 25\ e^{5k} \Longleftrightarrow \frac{6}{5} = e^{5k}$

$N(x) = 25\ e^{15k} = 25(e^{5k})^3 = 25\ (\frac{6}{5})^3 = \frac{216}{5} = 43.2$

Expected population in 1965 is 43,200 or 43,000, approximately.

23. a) $f(t) = e^{-t/RC} = e^{-1/0.1} = e^{-10} = (e^{-5})^2 \approx (0.0067)^2$

$\approx 4.49 \times 10^{-5}$

b) $\frac{-t}{RC} = \frac{-12 \times 10^{-4}}{48 \times 25 \times 10^{-6}} = -1$, hence

$f(t) = e^{-1} \approx 0.3679$

24. $\ln A = \frac{-t}{RC} \Longleftrightarrow t = -RC \ln A$

a) $t = -10^{-3} \ln 1 = 0$

b) $t = -(25 \times 10^3)(6.0 \times 10^{-4}) \ln 0.5 = -15 \ln 0.5$

$t \approx (-15)(-0.7) = 10.5$

25. $f(x) = e^x - x^3 + 3x$

$f'(x) = e^x - 3x^2 + 3$

Since $f(-0.3) \approx 0.7408 + 0.027 - 0.9 = -0.1322$

and $f(-0.2) \approx 0.8187 + 0.008 - 0.6 = 0.2267$

$f(x) = 0$ has a root between -0.3 and -0.2

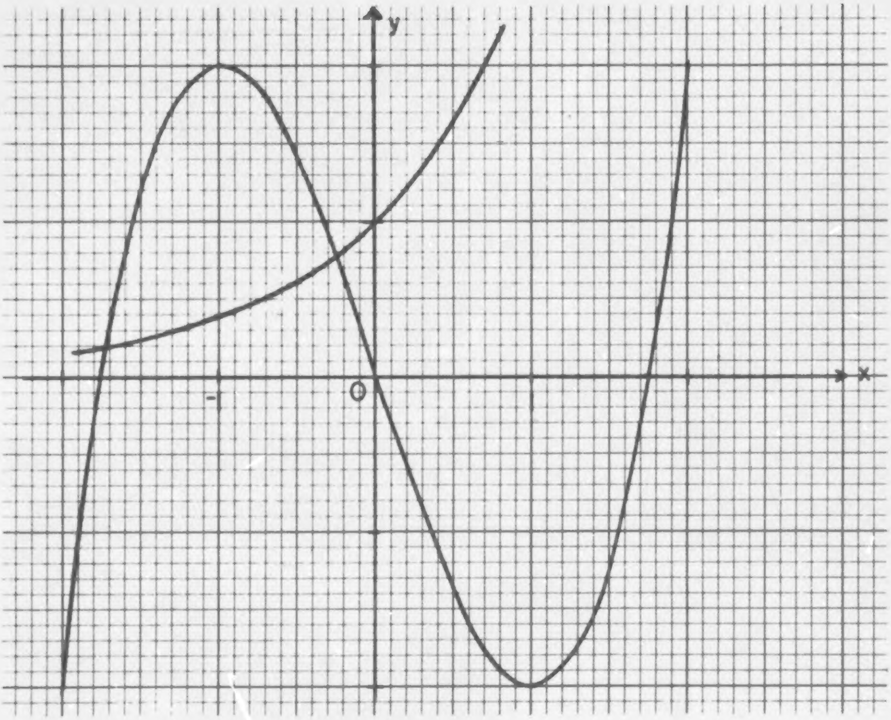
Let $x_1 = -0.3$, Then since $f'(-0.3) \approx 3.4708$,

$x_2 = -0.3 - \frac{-0.1322}{3.4708} \approx -0.262.$

Since $f(-0.26) \approx 0.0086$ and $f'(-0.26) \approx 3.57$,

$x_3 \approx -0.26 - \frac{0.0086}{3.57} \approx -0.26 - 0.0024 = -0.262.$

The required root is -0.26 correct to two decimals.

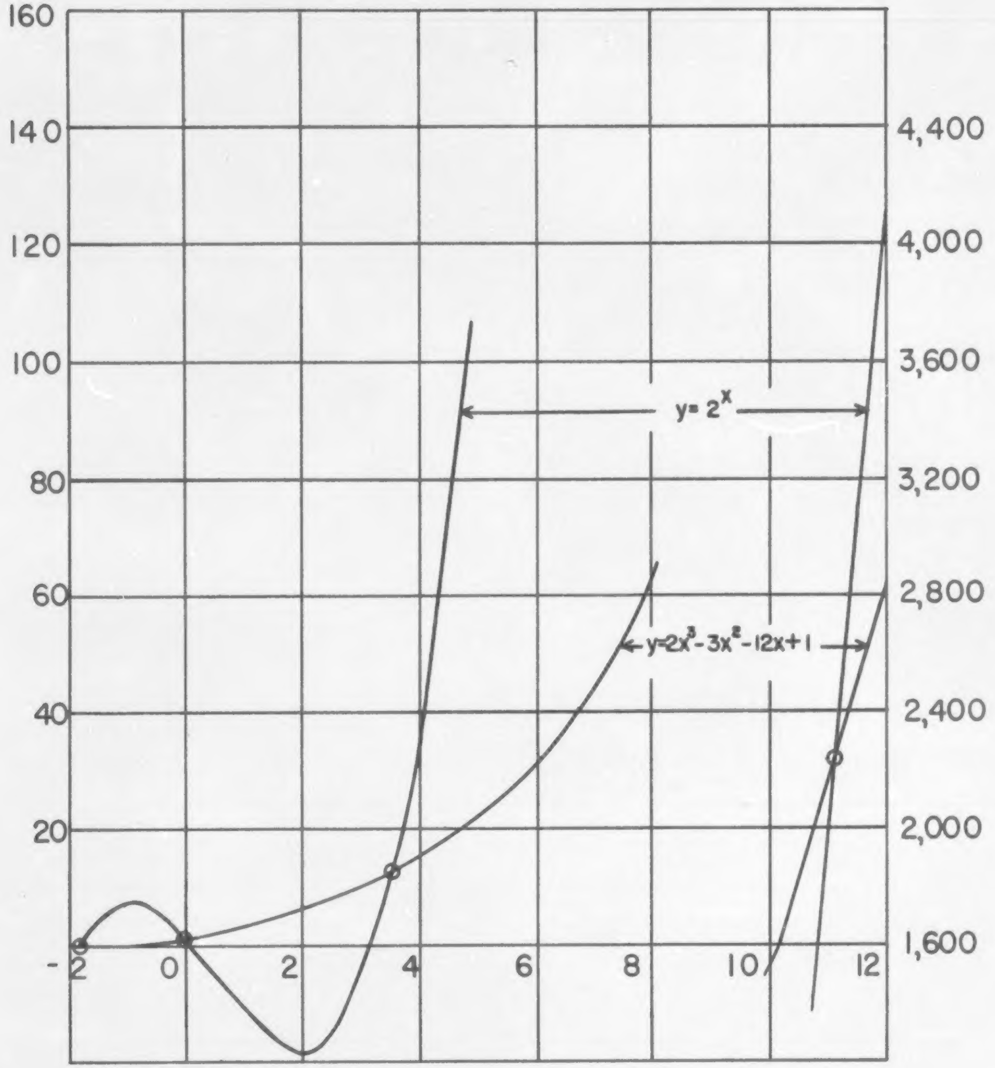


Graphs of $x \longrightarrow e^x$ and

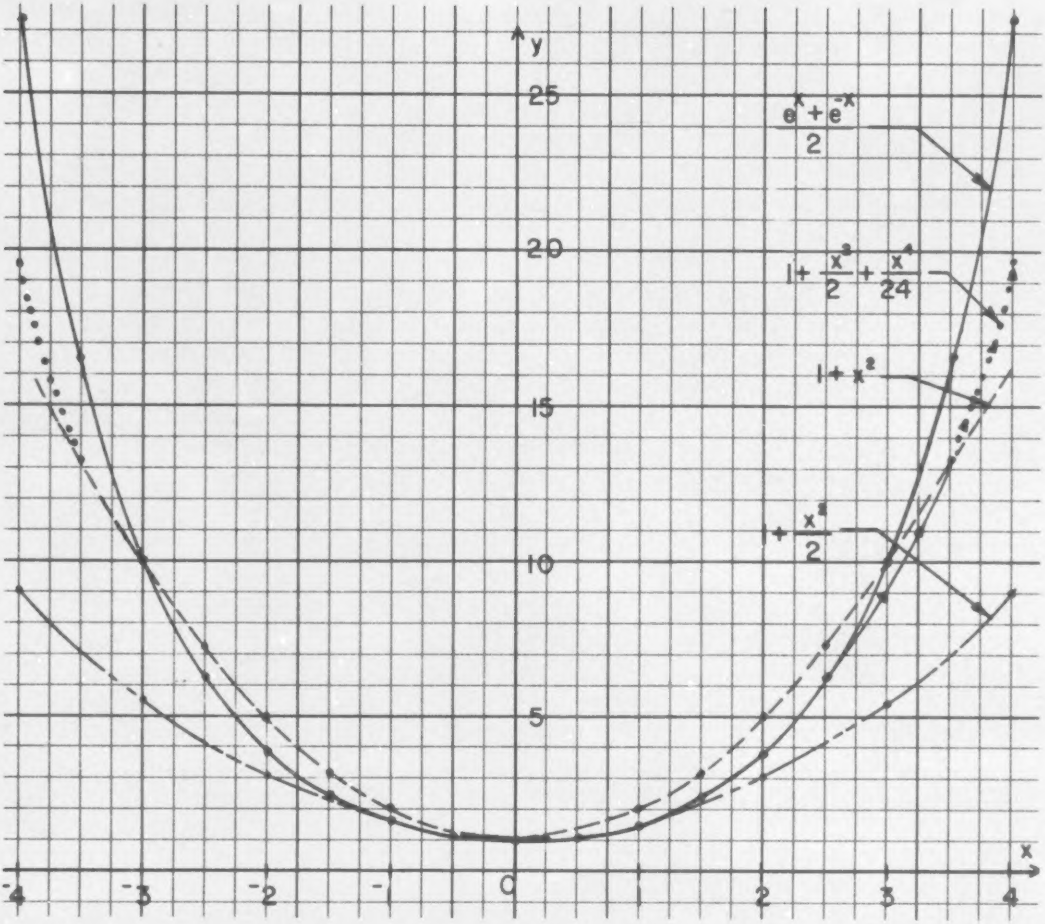
$x \longrightarrow x^3 - 3x$

Exercise 25

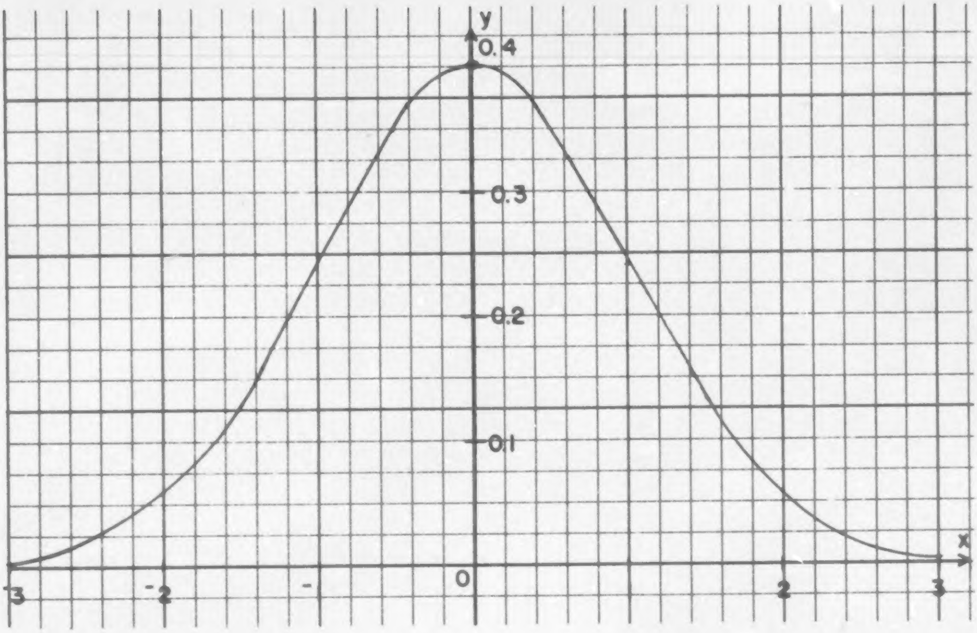
26. a) Graph
    b) three
    c) three
    d) four, four



27. a) See graph b) $y = 1+x^2$ c) See graph



28. a) See graph



c) Scores between 200 and 300: 2 %
Scores between 300 and 400: 14 %
Scores between 400 and 500: 34 %
Scores between 500 and 600: 34 %
Scores between 600 and 700: 14 %
Scores between 700 and 800: 2 %

Illustrative Test Questions

1. Simplify

   a) $e^{\ln x}$ b) $\ln(e^x)$

   for x an arbitrary positive real number.

2. If $\ln a = 3$ and $\ln b = 2$, find

   a) $\ln ab$ d) $\frac{\ln a}{\ln b}$

   b) $\ln a \cdot \ln b$ e) $\ln a^2$

   c) $\ln \frac{a}{b}$ f) $(\ln a)^2$

3. Solve the following equations for x.

   a) $x^{\log_2 5} = 5$ c) $3^{\ln x} = 3$

   b) $\log_7 x^3 = 3$ d) $2 \ln e^x = \frac{1}{2}$

4. Express each of the following as an exponential equation and solve for x.

   a) $\log_8 128 = x$ b) $\log_{1000} 10 = x$ c) $\log_{0.1} 0.01 = x$

5. Without graphing describe the relation between the graph of $f: x \longrightarrow a^x$ and the graph of $g: x \longrightarrow a^{-x}$.

6. Solve for x

   $\log_2(x - 2) + \log_2 x = 3.$

7. Are $f: x \longrightarrow \ln x^2$ and $g: x \longrightarrow 2 \ln x$ the same function? Explain.

8. What is the relationship between the graphs of $y = \ln x$ and $y = \ln kx$ $(k > 1)$?

Answers to Illustrative Test Questions

1. a) $e^{\ln x} = x$ b) $\ln(e^x) = x$

2. a) $\ln ab = 3 + 2 = 5$ d) $\frac{\ln a}{\ln b} = \frac{3}{2}$

   b) $\ln a \cdot \ln b = 3 \cdot 2 = 6$ e) $\ln a^2 = 2 \ln a = 6$

   c) $\ln \frac{a}{b} = 3 - 2 = 1$ f) $(\ln a)^2 = 3^2 = 9$

3. a) $x = 2$

   b) $x = 7$

   c) $\ln x = 1$, hence $x = e$

   d) $2 \ln e^{x} = \ln(e^{x})^{2} = \ln (e^{2x}) = \frac{1}{2} \Longleftrightarrow e^{\frac{1}{2}} = e^{2x}$

   and $x = \frac{1}{4}$

4. a) $\frac{7}{3}$ b) $\frac{1}{3}$ c) 2

5. The graphs are symmetric with respect to the y-axis.

6. $\log_2(x-2)(x) = 3 = \log_2 8 \Longleftrightarrow x^2 - 2x = 8$.

   The only solution is 4 since $x > 0$.

7. No. The domain of f is $\{x: x \neq 0\}$ and the domain of g is $\{x: x > 0\}$.

8. Since $\ln kx = \ln k + \ln x$ $(k > 1, x > 0)$, the two graphs have exactly the same shape, but the graph of $y = \ln kx$ is $\ln k$ units higher than the graph of $y = \ln x$.

Chapter 5

CIRCULAR FUNCTIONS

Introduction

This chapter is not a course in trigonometry in the solution-of-triangles sense. It is expected that, normally, this aspect of trigonometry will have been studied previous to the present chapter. This is not unconditionally necessary, however. If the student is unfamiliar with the simpler methods for solving right and oblique triangles, it is suggested that some time be devoted to Section 5-15, in the Appendices. If necessary, this work can be assigned concurrently with the study of Chapter 5. A further section in the Appendices (Section 5-16) includes practice material in proving trigonometric identities and solving trigonometric equations. Although some attention is given to these topics in the text (Sections 5-9 and 5-13), the treatment is probably insufficient if these subjects have not been studied before.

Emphasis is placed on the periodic property of sin and cos. (A relatively small part of the Chapter is devoted to tan.) Extensive use is made of the idea of rotating the plane about a perpendicular to it through the origin. This gives a certain unity to the discussion. Consistent with this emphasis, we have derived the formulas for $\sin(x + y)$ and $\cos(x + y)$ in terms of the simplest properties of rotation. We believe that this approach is a natural one (no tricks!) and that the student will really understand analytic trigonometry when he studies it in this way.

If time does not permit doing the whole chapter, Section 5-12 can well be omitted without loss of continuity.

5-1. Circular Motions and Periodicity

The emphasis throughout is on the periodic properties of the circular functions, i.e., the sine and cosine. In beginning the chapter you should emphasize that we shall talk here about functions which differ from those we have previously studied in that they have the property of periodicity.

One good way to visualize a periodic function is in terms of the machine developed in Section 1-1. If the function depicted by the machine is periodic, then when x, $x + a$, $x + 2a$, ..., $x + na$, are dropped into the hopper we obtain the same output, $f(x)$, in each case. In the next section we speak of laying rectangles containing one complete cycle of the function end to end and you may wish to use the idea here in order to illustrate further the meaning of periodicity.

The use of the uv - and xy - planes which we employ may be a source of difficulty at first. We wish to talk about the unit circle with which we define sin and cos, but later we shall need to display the graphs of $y = \sin x$ and $y = \cos x$ on an xy - plane. Since we are using x for arc length (to obtain the familiar $\sin x$ and $\cos x$) it might be confusing to teach the student to visualize x as both the horizontal axis on the plane of the unit circle and at the same time a length of circular arc. We feel that if care is exercised at the time the transition is made in Section 5-2, the use of u and v is more satisfactory than trying to get x to wear two hats in this section.

A more exact way of defining sin and cos is by a composition of two functions, one from the set of real numbers to the set of geometric points on the unit circle and the other from the set of points on the circle to the set of real numbers. Thus, if $x \in R$ and if P is a point on the unit circle, we have a function

$$f: x \longrightarrow P$$

and another function

$$g: P \longrightarrow \cos x$$

from which

$$gf: x \longrightarrow \cos x$$

and similarly for the sine. We feel, however, that the way in which we have handled it in the text, while possibly less rigorous, is certainly easier to teach and is perfectly adequate for our purposes.

The fact that cos and sin are functions from real numbers to real numbers should be emphasized. You might point out to the student that nowhere in this section have we used an angle and, although we have used the concept of arc length, sine and cosine are completely divorced from any geometric considerations. They are functions on the set of real numbers in the same sense as polynomials, say, or exponential functions. Too often when we speak of $\sin A$, the students feel that A must be an angle. Sometimes they think of A as being the degree measure or radian measure of an angle, but the idea that A need have no connection with an angle is usually very strange.

Exercises 5-1

The exercises lean on the notion of periodicity. The first five are not difficult. We have starred Exercises 6 - 9 since they require more insight than the others, but if Problem 7a is not assigned as homework, it should be covered in class, since this relationship is used in Section 5-4.

Answers to Exercises 5-1

1. The rotation of the earth about the sun every $365\frac{1}{4}$ days.

   The phases of the moon; period is about $29\frac{1}{2}$ days.

   The swinging of the pendulum of a clock; for a grandfather's clock, the period is usually 2 seconds.

The oscillation of a piston in a steam engine or internal combustion engine, period depends upon speed of engine.

The alternation of A.C. electric current; for 60 cycle current, the period is 1/60 second.

Oscillation of vacuum tubes, vibrations of strings of musical instruments (sound waves in general), etc.

2. a) $P(-\frac{\pi}{2}) = P(\frac{3\pi}{2} - 2\pi) = P(\frac{3\pi}{2})$;

   b) $P(3\pi) = P(\pi + 2\pi) = P(\pi)$;

   c) $P(-\frac{3\pi}{2}) = P(\frac{\pi}{2} - 2\pi) = P(\frac{\pi}{2})$;

   d) $P(4076\pi) = P(0 + 2038\cdot 2\pi) = P(0)$.

3. a) $(0, -1)$; c) $(0, 1)$;
   b) $(-1, 0)$; d) $(1, 0)$.

4. a) $x = \frac{3\pi}{2}, \frac{7\pi}{2}$ c) $x = 0, 2\pi$

   b) $x = \pi, 3\pi$ d) $x = \pi, 3\pi$

5. a) $x = \frac{\pi}{4}, \frac{5\pi}{4}$ b) $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

*6. a) $\sin 2x = \sin(2x + 2\pi)$ from periodicity of sin,
$= \sin 2(x + \pi)$, and the period is π.

   b) $\sin \frac{1}{2}x = \sin\left(\frac{1}{2}x + 2\pi\right)$

$= \sin \frac{1}{2}(x + 4\pi)$, and the period is 4π.

   c) $\cos 4x = \cos(4x + 2\pi)$

$= \cos 4(x + \frac{\pi}{2})$, and the period is $\frac{\pi}{2}$.

   d) $\cos \frac{1}{2}x = \cos (\frac{1}{2}x + 2\pi)$

$= \cos \frac{1}{2}(x + 4\pi)$, and the period is 4π.

*7. a) $f(x) = f(x + a), \quad g(x) = g(x + a).$ Given.

$f(x) + g(x) = f(x + a) + g(x + a).$ Addition Axiom.

$(f + g)(x) = (f + g)(x + a).$ By definition.

$\therefore f + g$ is periodic with period a. By definition.

To show that a is not necessarily the fundamental period, you can use, for example,

$$f: x \longrightarrow \ln \sin x \quad \text{and} \quad g: x \longrightarrow \ln \cos x,$$

each of which has period 2π. But

$$(f + g)(x) = \ln \sin x + \ln \cos x = \ln (\sin x \cos x)$$
$$= \ln (\tfrac{1}{2} \sin 2x)$$

and $f + g$ therefore has fundamental period π.

An even more striking example is afforded by

$$f: x \longrightarrow \sin x \quad \text{and} \quad g: x \longrightarrow -\sin x;$$

then $f + g: x \longrightarrow 0$ and has every real number as a period, but has no fundamental period.

b) $f(x) \cdot g(x) = f(x + a) \cdot g(x + a)$ Multiplication Axiom.

$(f \cdot g)(x) = (f \cdot g)(x + a)$ Definition.

$\therefore f \cdot g$ is periodic with period a. Definition.

An example in which a is not the fundamental period is

$$f: x \longrightarrow \sin x \quad \text{and} \quad g: x \longrightarrow \cos x$$

which yields

$$f \cdot g: x \longrightarrow \sin x \cos x = \frac{1}{2} \sin 2x$$

with fundamental period π.

*8. $f(x) = f(x + a)$ Given.

$g(x) = g(x)$ if g is defined at x.

$g(f(x)) = g(f(x + a))$ Substitution.

$(gf)(x) = (gf)(x + a)$ By definition of gf.

$\therefore gf$ is periodic with period a.

*9. If a is a period of cos, it must be true that

$$\cos(x + a) = \cos x$$

for all $x \in R$. In particular, it must be true if $x = 0$:

$$\cos a = \cos 0 = 1.$$

But the only point on the unit circle with abscissa 1 is (1, 0), which corresponds to $x = 0 + 2n\pi$.

The proof for sin is similar; use $x = \frac{\pi}{2}$.

5-2. Graphs of Sine and Cosine

The rectangle device used here can be a very useful one in teaching the student to graph periodic functions. By establishing the period and amplitude visually it directs his attention to a specific region of the plane with respect to both the domain and range of the function.

We use the geometric argument to obtain specific values of the functions, because it is the simplest and most familiar tool available to the student. We hope that you will emphasize the symmetric nature of the unit circle and that the student will be encouraged to use considerations of symmetry whenever possible.

Exercises 5-2

The exercises develop some simple symmetry properties of sin x and cos x, and lead the student into understanding the effect of the constants A, B, and C in $y = A\sin(Bx + C)$.

Answers to Exercises 5-2

1. a) $f(3\pi) = f(\pi) = -1$ d) $f(\frac{25\pi}{6}) = f(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

b) $f(\frac{7\pi}{3}) = f(\frac{\pi}{3}) = \frac{1}{2}$ e) $f(-7\pi) = f(\pi) = -1$

c) $f(\frac{9\pi}{2}) = f(\frac{\pi}{2}) = 0$ f) $f(-\frac{10\pi}{3}) = f(\frac{2\pi}{3}) = -\frac{1}{2}$

2. a) $f(\pi) = 0$

   b) $f(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$

   c) $f(\frac{\pi}{2}) = 1$

   d) $f(\frac{\pi}{6}) = \frac{1}{2}$

   e) $f(\pi) = 0$

   f) $f(\frac{2\pi}{3}) = \frac{\sqrt{3}}{2}$.

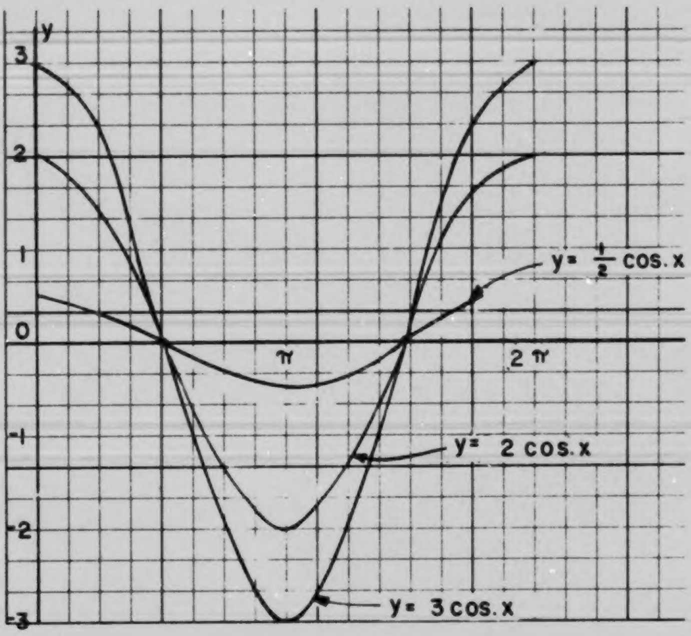
3. a) $x = \frac{\pi}{4} + n\pi$

   b) $x = \frac{3\pi}{4} + n\pi$

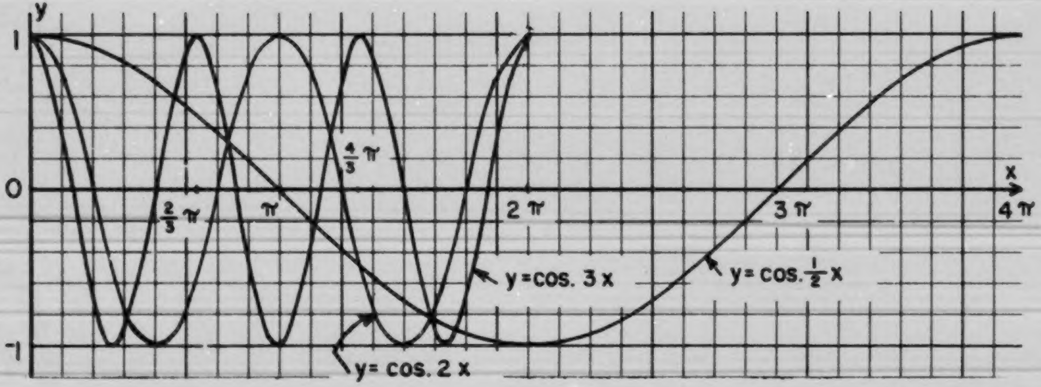
   c) $x = n\pi$

   d) For all values of x.

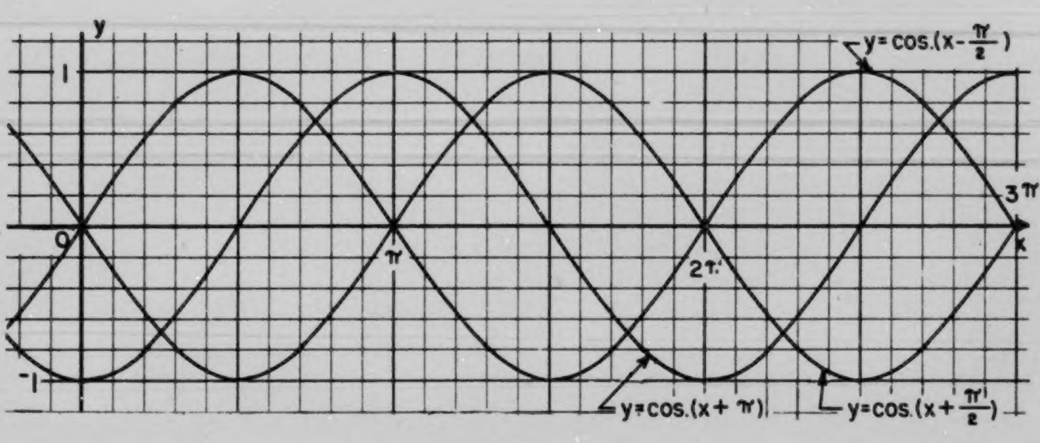
4.



5.


y
1
0
-1
x
$\frac{2}{3}\pi$
π
$\frac{4}{3}\pi$
2π
3π
4π
y = cos. 3 x
y = cos. $\frac{1}{2}$ x
y = cos. 2 x

6.

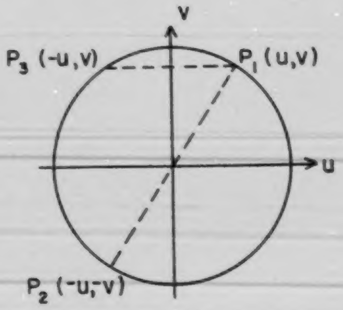

y
1
0
-1
x
π
2π
3π
y = cos.(x − $\frac{\pi}{2}$)
y = cos.(x + π)
y = cos.(x + $\frac{\pi}{2}$)

7. a) The values of the ordinate are multiplied by k.

   b) The period of the graph is $\frac{2\pi}{k}$.

   c) The graph is shifted to the left by the amount $x = k$.

8. $\cos(x - \frac{\pi}{2}) = \sin x$.



9. a) P_1 and P_2 are symmetric with respect to the origin.

   $P_1 = \rho(x) = (u, v)$,

   $P_2 = \rho(x - \pi) = \rho(x + \pi)$
   $= (-u, -v)$.

   Hence, $\cos x = -\cos(x - \pi) =$
   $-\cos(x + \pi)$,

   and
   $\sin x = -\sin(x - \pi) =$
   $-\sin(x + \pi)$.

   b) P_1 and P_3 are symmetric with respect to the v-axis.

   $P_1 = \rho(x) = (u, v)$,

   $P_2 = \rho(-x - \pi) = \rho(-x + \pi) = (-u, v)$.

   Hence, $\cos x = -\cos(-x - \pi) = -\cos(-x + \pi)$,

   and $\sin x = \sin(-x - \pi) = \sin(-x + \pi)$.

5-3. Angle and Angle Measure

This is probably review material for most students at this level. Formulas (1) and (2) are the standard radian-degree relationships and the exercises are routine drill in going from one to the other.

Answers to Exercises 5-3

1. a) 120°

b) 30°

c) -120°

d) 210°

e) 360°

f) 150°

g) 480°

h) 648°

i) 585°

2. a) $\frac{3\pi}{2}$

b) $-\frac{\pi}{6}$

c) $\frac{3\pi}{4}$

d) $\frac{8\pi}{3}$

e) $\frac{13\pi}{12}$

f) $-\frac{7\pi}{12}$

g) $\frac{9\pi}{2}$

h) $\frac{19\pi}{18}$

i) $\frac{\pi}{10}$

3. $\alpha = \frac{2A}{r^2} = \frac{2\cdot 9\pi}{9} = 2\pi$

4. $A = \frac{\alpha\, r^2}{2} = \frac{(3/2)\pi\cdot 4}{2} = 3\pi$ square units.

5. a) Since $90^\circ = 100$ "units", $1^\circ = \frac{10}{9}$ "units".

b) Since $\frac{\pi}{2} = 100$ "units", 1 radian $= \frac{200}{\pi}$ "units".

c) $\alpha = \frac{s}{r} = \frac{2r}{r} = 2$ radians; hence $\alpha = \frac{400}{\pi}$ "units".

5-4. Uniform Circular Motion

This unit should be taught with care, since the material included will be used in Section 5-8. In dealing with sin and cos as time functions we use ωt where ω is the angular velocity, because this is the form in which it appears in most scientific applications. Since up to this point we have dealt with functions connected with an arc length x, you should spend a little time familiarizing the student with ωt.

The device used in the text to visualize the behavior of a wave is only one of several which you may wish to try. Most currently available trigonometry texts have some such approach to the problem, and you should supplement the textual explanation with any other means you feel appropriate.

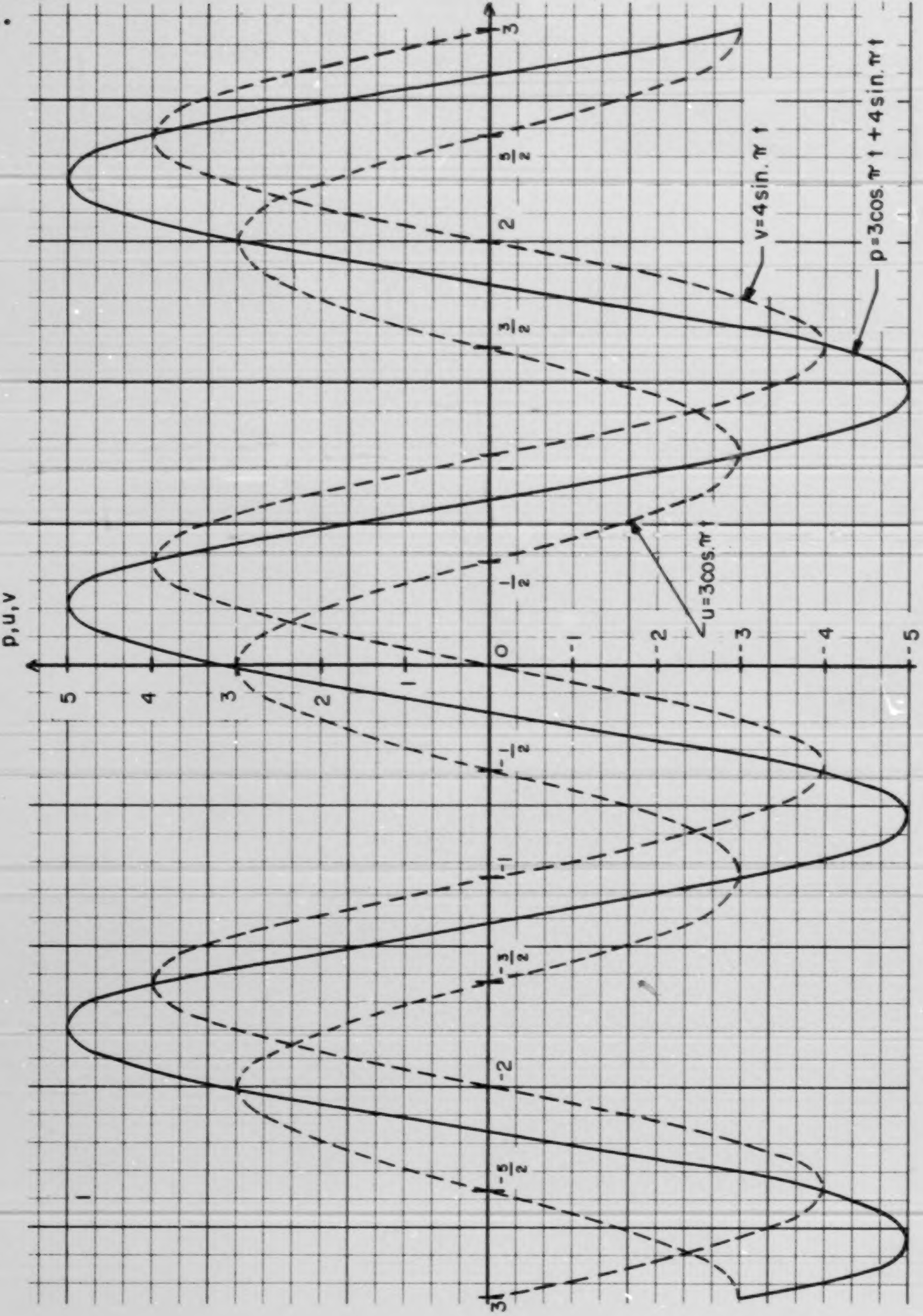
We chose the acoustical example to build upon since the addition of pressures is intuitively simple.

When we use a graph to enhance the student's understanding of a function which maps real numbers into real numbers, we give a true picture of the function only when we use the same scale on both axes. We have followed this practice in most of the graphs of this section of the text. On the other hand, it is sometimes desirable to distort the graph by using different scales in order to show important details which might otherwise be indistinct or confused, and when we graph an equation which describes the relationship between two physical quantities, the question of equal scales may be meaningless. If the pressure p at time t is given by an equation of the form $p = P \cos (\omega t + \alpha)$, we cannot use the same scale on the p-axis as on the t-axis because there is no common measure for time and pressure. Because this situation is one of common occurrence in applications of the circular functions, we have not always insisted on the equal-scales principle. See, for example, Figure 5-4e, and many of the graphs in this section of the commentary.

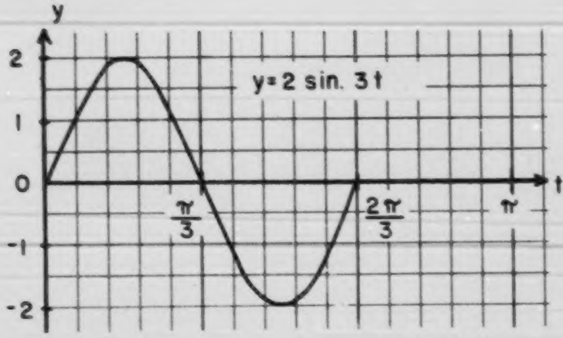
Answers to Exercises 5-4

1. See Graph. (Note scales.)
   The graph of $p = 3 \cos \pi t + 4 \sin \pi t$ is periodic, with period 2, since corresponding points on the graph are $2, 4, 6, \ldots, 2n$ units apart when measured along the t-axis. (Periods of $3 \cos \pi t$ and $4 \sin \pi t$ same as period of p.)

1.


p,u,v
t
5
4
3
2
1
0
-1
-2
-3
-4
-5
3
5/2
2
3/2
1/2
-1/2
-1
-3/2
-2
-5/2
-3
v=4sin. πt
p=3cos. πt + 4sin. πt
u=3cos. πt

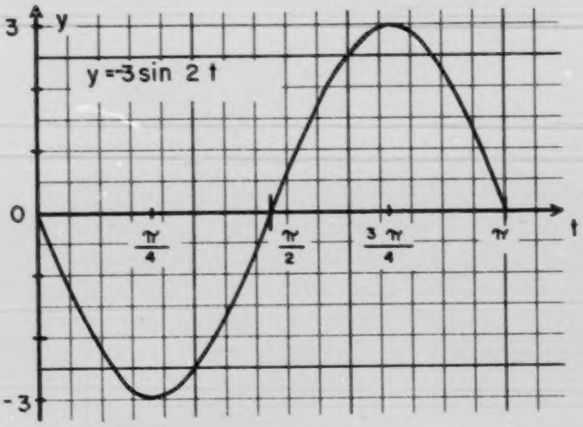
2. a)



The period is $\frac{2\pi}{3}$.

The range is $-2 \leq y \leq 2$.

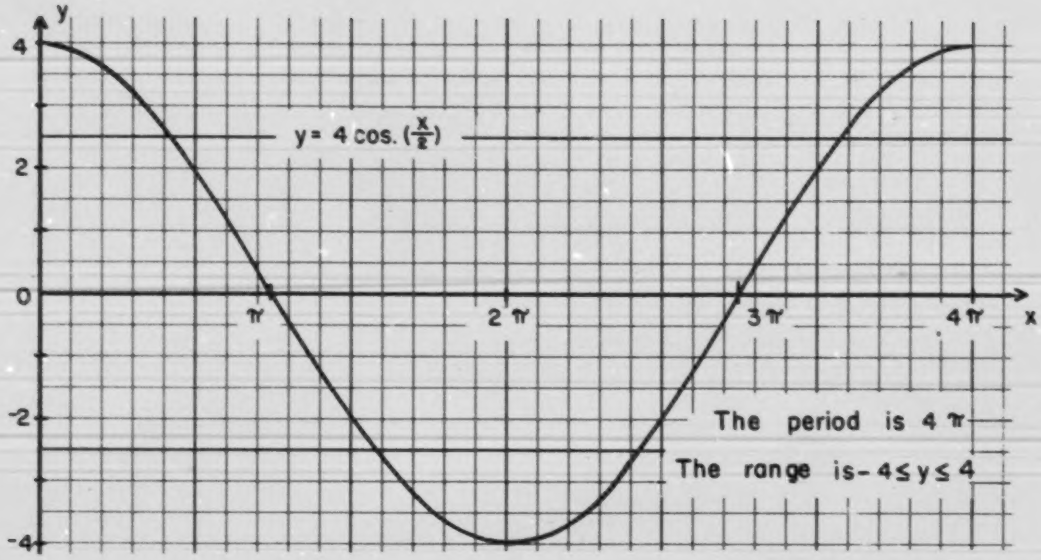
b)



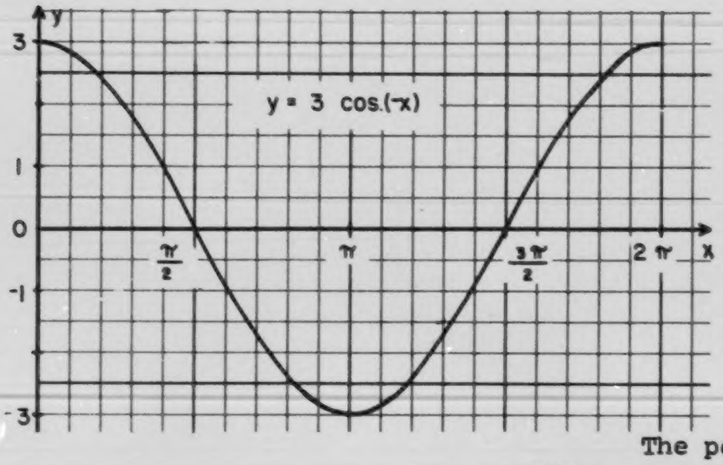
The period is π.

The range is $-3 \leq y \leq 3$.

c)



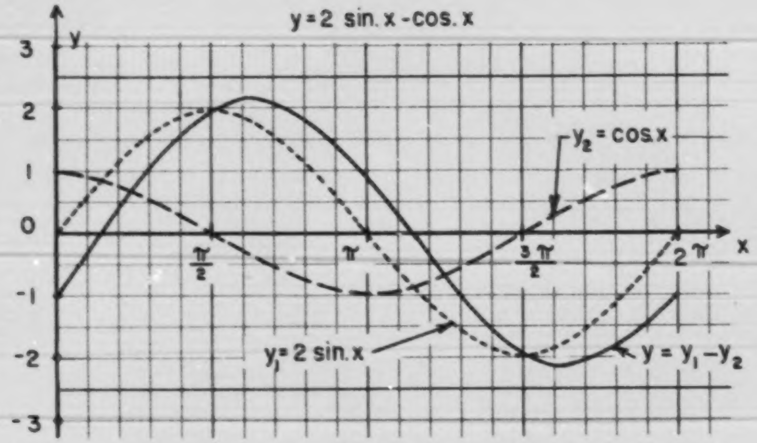
d)



The period is 2π.

The range is $-3 \le y \le 3$.

e)



The period is 2π. The range is $-\sqrt{5} \leq y \leq \sqrt{5}$.

5-5. Vectors and Rotations

We chose the vector approach to the addition formulas for two reasons. First, it should be a means of deriving these relationships different from any which the student has previously encountered. Second, it is an extremely simple and efficient means of obtaining these relationships. We do not, of course, intend this to be a thorough treatment of vectors.

We anticipate that the idea of a rotation as a function, and its effect on a vector, will have to be explained very carefully. You should do a lot of blackboard work here, giving a variety of simple manipulative illustrations. By using chalk of different colors, you can probably improve on some of the figures (such as Figure 5-5e, for example) in the book. Show vectors rotated in both directions; illustrate rotations followed by rotations; show the rotations of the components of the vector as the vector rotates; in general, make sure that the ideas involved and the symbolism expressing the ideas are clear.

Exercises 5-5

You may wish to devise additional drill exercises in the use of rotation. Exercises 1, 2, 3 and 4 are cases in point and such problems are easy to make up.

Answers to Exercises 5-5

1. $T = (\frac{\sqrt{2}}{2})U + (\frac{\sqrt{2}}{2})V$, $x = \frac{\pi}{4}$.

2. a) $T = (-\frac{1}{2})U + (-\frac{\sqrt{3}}{2})V$, $x = \frac{4\pi}{3}$.

   b) $T = (-\frac{\sqrt{3}}{2})U + (-\frac{1}{2})V$, $x = \frac{7\pi}{6}$.

3. a) $f(U) = (0)U + (-1)V = -V$

   b) $f(U) = U$

4. a) $f(U) = (\frac{\sqrt{2}}{2})U + (\frac{\sqrt{2}}{2})V$

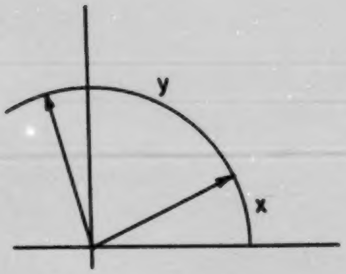
   b) $f(U) = (\frac{1}{2})U + (\frac{\sqrt{3}}{2})V$

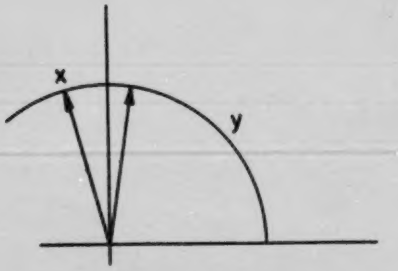
5. $f(U) = (-\frac{\sqrt{2}}{2})U + (\frac{\sqrt{2}}{2})V$

6. $\frac{3\pi}{4} = \frac{\pi}{4} + \frac{\pi}{2}$, and the result follows from (7).

7. Let f correspond to a rotation x and g to a rotation y. Then x and y are real numbers, hence $y + x = x + y$, and the desired result follows.

   Geometrically, the result means that a rotation through arc x followed by a rotation through arc y is equivalent to a rotation through arc y followed by a rotation through arc x.





8. Since $V = g(U)$,

$$\begin{aligned} f(V) &= f(g(U)) = (fg)(U) \\ &= (gf)(U), \text{ by the result of Exercise 7.} \end{aligned}$$

9. From Exercise 8,

$$\begin{aligned} f(V) &= f(g(U)) \\ &= g(f(U)) \\ &= g(uU + vV) \\ &= ug(U) + vg(V) \\ &= uV - vU \end{aligned}$$

since

$$g(U) = V \quad \text{and} \quad g(V) = -U.$$

5-6. Addition Formulas for Sine and Cosine

The derivation of $\cos (x + y)$ and $\sin (x + y)$ is usually accomplished either by geometric considerations in the first quadrant (which then involve a great deal of work to generalize), or by use of the distance formula. As remarked before, we feel the vector approach to be new and instructive and, in essence, simpler than either of the aforementioned. We include the page on the relation to complex numbers to show still another means of deriving these formulas.

Exercises 5-6

The exercises are, in general, identities; applications of the sum and difference formulas. You may wish to illustrate a few samples on the blackboard before asking the students to work the exercises. Exercises 4, 5 and 6 are important since the tangent function appears here for the first time and some of its properties are investigated. You should be sure to cover these exercises at some point in the work.

Answers to Exercises 5-6

1. a) $\cos(\frac{\pi}{2} - x) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$

$= 0 + \sin x$

$= \sin x$

b) $\sin(\frac{\pi}{2} - x) = \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x$

$= \cos x - 0$

$= \cos x$

c) $\cos(x + \frac{\pi}{2}) = \cos x\cos\frac{\pi}{2} - \sin x\sin\frac{\pi}{2}$

$= 0 - \sin x$

$= -\sin x$

d) $\sin(x + \frac{\pi}{2}) = \sin x\cos\frac{\pi}{2} + \cos x\sin\frac{\pi}{2}$

$= 0 + \cos x$

$= \cos x$

e) $\cos(\pi - x) = \cos\pi\cos x + \sin\pi\sin x$

$= (-1)\cos x + 0$

$= -\cos x$

f) $\sin(\pi - x) = \sin\pi\cos x - \cos\pi\sin x$

$= 0 - (-1)\sin x$

$= \sin x$

g) $$\begin{aligned}\cos\left(\frac{3\pi}{2}+x\right) &= \cos\frac{3\pi}{2}\cos x - \sin\frac{3\pi}{2}\sin x\\ &= 0 - (-1)\sin x\\ &= \sin x\end{aligned}$$

h) $$\begin{aligned}\sin\left(\frac{3\pi}{2}+x\right) &= \sin\frac{3\pi}{2}\cos x + \cos\frac{3\pi}{2}\sin x\\ &= (-1)\cos x + 0\\ &= -\cos x\end{aligned}$$

i) $$\begin{aligned}\sin\left(\frac{\pi}{4}+x\right) &= \sin\frac{\pi}{4}\cos x + \cos\frac{\pi}{4}\sin x\\ &= \left(\frac{\sqrt{2}}{2}\right)(\cos x + \sin x);\\ \cos\left(\frac{\pi}{4}-x\right) &= \cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x\\ &= \left(\frac{\sqrt{2}}{2}\right)(\cos x + \sin x).\end{aligned}$$

Hence, $\sin\left(\frac{\pi}{4}+x\right) = \cos\left(\frac{\pi}{4}-x\right)$.

2. $$\begin{aligned}\sin(x-y) &= \sin[x+(-y)]\\ &= \sin x\cos(-y) + \cos x\sin(-y)\\ &= \sin x\cos y - \cos x\sin y\end{aligned}$$

*3. Formula 10: $\cos(x-y) = \cos x\cos y + \sin x\sin y$

To derive 7: $$\begin{aligned}\cos(x+y) &= \cos[x-(-y)]\\ &= \cos x\cos(-y) + \sin x\sin(-y)\\ &= \cos x\cos y - \sin x\sin y\end{aligned}$$

To derive 8: $$\begin{aligned}\sin(x+y) &= \cos\left[\frac{\pi}{2}-(x+y)\right] \text{ from Exercise 1a.}\\ &= \cos\left[\left(\frac{\pi}{2}-x\right)-y\right]\\ &= \cos\left(\frac{\pi}{2}-x\right)\cos y + \sin\left(\frac{\pi}{2}-x\right)\sin y\end{aligned}$$

To simplify $\cos\left(\frac{\pi}{2}-x\right)$ and $\sin\left(\frac{\pi}{2}-x\right)$, use Exercise 1a.

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right] = \cos x.$$

Hence, $\cos\left(\frac{\pi}{2} - x\right)\cos y + \sin\left(\frac{\pi}{2} - x\right)\sin y$

becomes $\sin x \cos y + \cos x \sin y$.

Therefore, $\sin (x + y) = \sin x \cos y + \cos x \sin y$.

To derive 11, use 8 just obtained:

$$\begin{aligned}\sin (x - y) &= \sin [x + (-y)]\\ &= \sin x \cos (-y) + \cos x \sin (-y)\\ &= \sin x \cos y - \cos x \sin y.\end{aligned}$$

4. $\tan : x \to \frac{\sin x}{\cos x}\ \left(x \neq \pm \frac{\pi}{2} + 2n\pi\right)$

To prove that tan is periodic with period π, we must prove that $\tan (x + \pi) = \tan x$.

From the definition, $\tan (x + \pi) = \frac{\sin (x + \pi)}{\cos (x + \pi)}$

$= \frac{-\sin x}{-\cos x}$ (from Exercise 5-2, 9a)

$= \tan x$

Now $\tan\left(\pm \frac{\pi}{2} + 2n\pi\right) = \dfrac{\sin\left(\pm \frac{\pi}{2} + 2n\pi\right)}{\cos\left(\pm \frac{\pi}{2} + 2n\pi\right)}$.

But the denominator of this fraction is zero and therefore the values of $\tan\left(\pm \frac{\pi}{2} + 2n\pi\right)$ are undefined.

5. $\tan (x \pm y) = \dfrac{\sin (x \pm y)}{\cos (x \pm y)}$

$$= \frac{\sin x \cos y \pm \cos x \sin y}{\cos x \cos y \mp \sin x \sin y}$$

Dividing numerator and denominator by $\cos x \cos y$,

$$= \frac{\dfrac{\sin x \cos y}{\cos x \cos y} \pm \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} \mp \dfrac{\sin x \sin y}{\cos x \cos y}}$$

$$= \frac{\dfrac{\sin x}{\cos x} \pm \dfrac{\sin y}{\cos y}}{1 \mp \dfrac{\sin x}{\cos x} \cdot \dfrac{\sin y}{\cos y}}$$

$$= \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

6. $\tan(\pi - x) = \dfrac{\tan \pi - \tan x}{1 + \tan \pi \tan x}$

$$= \frac{0 - \tan x}{1 + 0}$$

$$= -\tan x$$

$\tan(\pi + x) = \dfrac{\tan \pi + \tan x}{1 - \tan \pi \tan x}$

$$= \frac{0 + \tan x}{1 - 0} = \tan x$$

$\tan(-x) = \dfrac{\sin(-x)}{\cos(-x)} = \dfrac{-\sin x}{\cos x} = -\tan x$

7. $\sin 2x = \sin(x + x) = \sin x \cos x + \cos x \sin x$

$$= 2 \sin x \cos x$$

$\cos 2x = \cos(x + x) = \cos x \cos x - \sin x \sin x$

$$= \cos^2 x - \sin^2 x$$

$\tan 2x = \tan(x + x) = \dfrac{\tan x + \tan x}{1 - \tan x \tan x}$

$$= \frac{2 \tan x}{1 - \tan^2 x}$$

8. $\sin 3x = \sin(2x + x)$

$$= \sin 2x \cos x + \cos 2x \sin x$$

$$= 2 \sin x \cos^2 x + (\cos^2 x - \sin^2 x) \sin x$$

$$= 3 \sin x \cos^2 x - \sin^3 x$$

9. $\cos 2x = 1 - 2\sin^2 x$

Let $x = \frac{y}{2}$

$\cos y = 1 - 2\sin^2 \frac{y}{2}$

$\sin^2 \frac{y}{2} = \frac{1 - \cos y}{2}$

$\sin \frac{y}{2} = \pm\sqrt{\frac{1 - \cos y}{2}}$

10. $\cos 2x = 2\cos^2 x - 1$

Let $x = \frac{y}{2}$

$\cos y = 2\cos^2 \frac{y}{2} - 1$

$\cos^2 \frac{y}{2} = \frac{1 + \cos y}{2}$

$\cos \frac{y}{2} = \pm\sqrt{\frac{1 + \cos y}{2}}$

11. $\tan \frac{y}{2} = \frac{\sin \frac{y}{2}}{\cos \frac{y}{2}}$

$= \pm\sqrt{\frac{1 - \cos y}{1 + \cos y}}$

Multiply the right member by $\sqrt{\frac{1 - \cos y}{1 - \cos y}}$;

$\tan \frac{y}{2} = \pm\sqrt{\frac{(1 - \cos y)^2}{1 - \cos^2 y}}$

$= \pm\sqrt{\frac{(1 - \cos y)^2}{\sin^2 y}}$

$= \frac{1 - \cos y}{\sin y}$

Note: The result given is correct since $\tan \frac{y}{2}$ and $\frac{1 - \cos y}{\sin y}$ agree in sign for all possible combinations of sign of $\cos y$ and $\sin y$.

Multiply the right member by $\sqrt{\dfrac{1 + \cos y}{1 + \cos y}}$;

$$\tan \frac{y}{2} = \pm\sqrt{\frac{1 - \cos^2 y}{(1 + \cos y)^2}}$$

$$= \pm\sqrt{\frac{\sin^2 y}{(1 + \cos y)^2}}$$

$$= \frac{\sin y}{1 + \cos y}$$ (See previous note.)

Alternatively,

$$\tan \frac{y}{2} = \frac{\sin \frac{y}{2}}{\cos \frac{y}{2}} = \frac{2 \sin \frac{y}{2} \cos \frac{y}{2}}{2 \cos^2 \frac{y}{2}} = \frac{\sin y}{1 + \cos y}$$

5-7. Construction and Use of Tables of Circular Functions

Since this material is largely in the nature of a review, you will probably not wish to spend much time on it. The table of decimal fractions of $\pi/2$ will be new to the student, but we use it as we do any other table and it should cause no difficulty.

Answers to Exercises 5-7a

1. Table I is not folded because the values of x are given in such a way that they are not symmetrical about

   $x = \frac{\pi}{4} \approx 0.785$. For example,

   $$\cos 0.60 = \sin (\tfrac{\pi}{2} - 0.60).$$

   Since we are using radian measure, $\frac{\pi}{2}$ is irrational, and hence we would have to use an irrational interval (as is done in Table II) to get a symmetric table.

   $$\cos 0.60 \approx \sin (1.57 - 0.60)$$

   $$\approx \sin 0.97.$$

From the table, $\cos 0.60 = 0.8253$ and $\sin 0.97 = 0.8249$. The values of $\cos 0.60$ and $\sin 0.97$ would have to be the same if the table could be folded.

2. a) $\sin 0.73 \approx 0.6669$, $\cos 0.73 \approx 0.7452$

b) $\sin(-5.17) = \sin(-5.17 + 2\pi)$

$\approx \sin 1.11 \approx 0.8957$

$\cos(-5.17) \approx \cos 1.11 \approx 0.4447$

c) $\sin 1.55 \approx 0.9998$, $\cos 1.55 \approx 0.0208$

d) $\sin 6.97 = \sin(6.97 - 2\pi)$

$\approx \sin 0.69 \approx 0.6365$

$\cos 6.97 \approx \cos 0.69 \approx 0.7712$

3. a) $\sin x \approx 0.1099$, $x \approx 0.11$

b) $\cos x \approx 0.9131$, $x \approx 0.42$

c) $\sin x \approx 0.6495$, $x \approx 0.71$

d) $\cos x \approx 0.5403$, $x \approx 1.00$

Note: Hereafter we use "=" for "≈".

4. a) $\sin 0.31(\frac{\pi}{2}) = 0.468$, $\cos 0.31(\frac{\pi}{2}) = 0.884$

b) $\sin 0.79(\frac{\pi}{2}) = 0.946$, $\cos 0.79(\frac{\pi}{2}) = 0.324$

c) $\sin 0.62(\frac{\pi}{2}) = 0.827$, $\cos 0.62(\frac{\pi}{2}) = 0.562$

d) $\sin 0.71(\frac{\pi}{2}) = 0.898$, $\cos 0.71(\frac{\pi}{2}) = 0.440$

5. a) $\sin \omega t = 0.827$, $t = 0.62$

b) $\cos \omega t = 0.905$, $t = 0.28$

c) $\sin \omega t = 0.475$, $t = 0.315$

d) $\cos \omega t = 0.795$, $t = 0.415$

6. a) $\sin 45^\circ = 0.707$, $\cos 45^\circ = 0.707$

   b) $\sin 73^\circ = 0.956$, $\cos 73^\circ = 0.292$

   c) $\sin 36.2^\circ = 0.591$, $\cos 36.2^\circ = 0.807$

   d) $\sin 81.5^\circ = 0.989$, $\cos 81.5^\circ = 0.148$

7. a) $\sin x = 0.629$, $x = 39^\circ$

   b) $\cos x = 0.991$, $x = 7.7^\circ$

   c) $\sin x = 0.621$, $x = 38.4^\circ$

   d) $\cos x = 0.895$, $x = 26.5^\circ$

Answers to Exercises 5-7b

1. $\sin 1.73 = \sin(\pi - 1.73) = \sin 1.41 = 0.9871$ (Table I)
2. $\cos 1.3\pi = -\cos(1.3\pi - \pi) = -\cos 0.3\pi = -\cos 0.60(\frac{\pi}{2})$
   $= -0.588$ (Table II)
3. $\sin(-.37) = -\sin .37 = -0.3616$ (Table I)
4. $\sin(-.37\pi) = -\sin .74(\frac{\pi}{2}) = -0.918$ (Table II)
5. $\cos 2.8\pi = \cos(2.8\pi - 2\pi) = \cos 0.8\pi = -\cos(\pi - 0.8\pi)$
   $= -\cos 0.2\pi = -\cos 0.4(\frac{\pi}{2}) = -0.809$ (Table II)
6. $\cos 1.8\pi = \cos(2\pi - 1.8\pi) = \cos 0.2\pi = 0.809$ (from Ex. 5)
7. $\cos 3.71 = -\cos(3.71 - \pi) = -\cos 0.57 = -0.8419$ (Table I)
8. $\sin 135^\circ = \sin(180^\circ - 135^\circ) = \sin 45^\circ = 0.707$ (Table III)
9. $\cos(-135^\circ) = -\cos(180^\circ - 135^\circ) = -\cos 45^\circ = -0.707$ (Table III)
10. $\sin 327^\circ = -\sin(360^\circ - 327^\circ) = -\sin 33^\circ = -0.545$ (Table III)
11. $\cos(-327^\circ) = \cos(360^\circ - 327^\circ) = \cos 33^\circ = 0.839$ (Table III)
12. $\cos 12.4\pi = \cos(12.4\pi - 12\pi) = \cos 0.4\pi = \cos 0.8(\frac{\pi}{2})$
    $= 0.309$ (Table II)

13. $\sin 12.4 = -\sin(4\pi - 12.4) = -\sin 0.16 = -0.1593$ (Table I)

*14. $\cos(\sin .3\pi) = \cos\left(\sin 0.6\left(\frac{\pi}{2}\right)\right) = \cos 0.809$ (Table II)

$= 0.6902$ (Table I)

*15. $\sin(\sin .7) = \sin 0.6442 = 0.6006$ (Table I)

5-8. Pure Waves. Frequency, Amplitude and Phase

We chose cos as our standard wave because its first peak occurs at 0. Since we are using peaks to discuss phase, cos serves better than sin which peaks first at $\pi/2$. The phase α is often called a lag ($\alpha > 0$) or a lead ($\alpha < 0$); if $\alpha = 0$, the wave is in phase with the standard. By using $0 \leq \alpha < 2\pi$, we avoid all mention of a wave "leading". This is a departure from the conventional, in that most sciences which have occasion to discuss lead or lag use both. You may wish to explore this idea by examining with the class the effect of using $-\pi \leq \alpha < \pi$, and show that $\alpha < 0$ represents a lead in the sense that $\alpha > 0$ represents a lag.

Note that α is not, in general, the abscissa of the first maximum. In fact, $A \cos(\omega t - \alpha)$ $(A > 0)$ reaches a peak when $\omega t - \alpha = 0$ or $t = \frac{\alpha}{\omega}$; hence the abscissa of the first maximum is $\frac{\alpha}{\omega}$.

Answers to Exercises 5-8

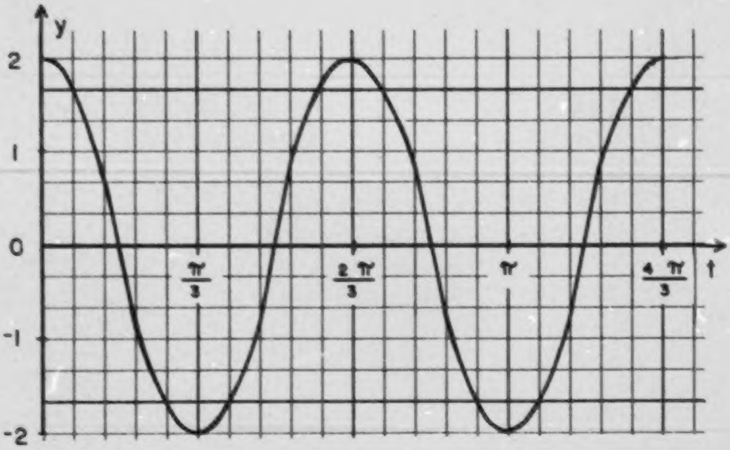
1. Since $p \approx 5 \cos(\pi t - 0.927)$,

$p = 0$ if $\pi t - 0.927 \approx \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

Hence, if $t \approx \frac{0.927}{\pi} + \frac{1}{2}$ or $\frac{0.927}{\pi} + \frac{3}{2}$.

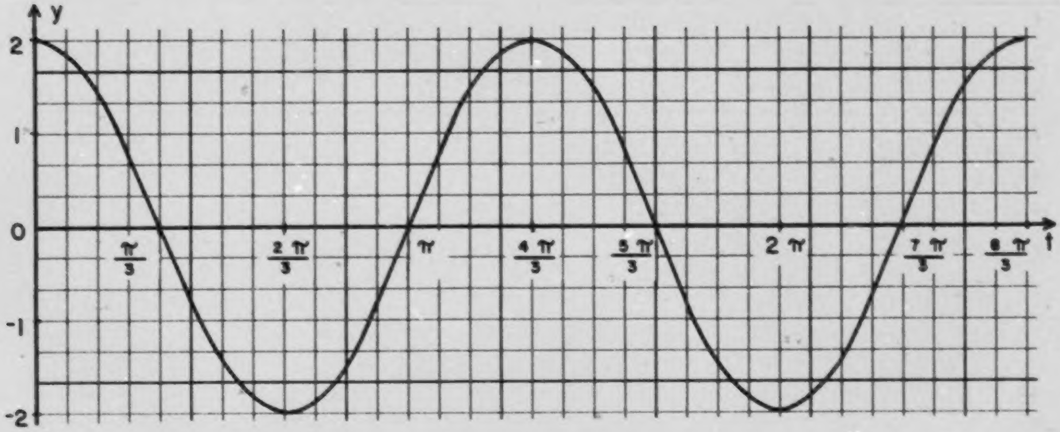
Since $\frac{0.927}{\pi} \approx 0.29$, $t \approx 0.79$ or 1.79.

2. a)



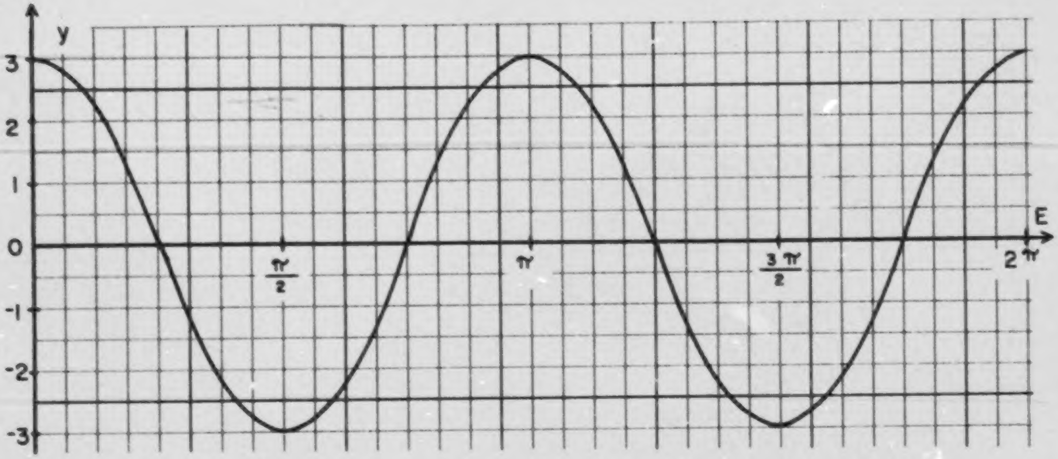
Amplitude = 2,
Period = $\frac{2\pi}{3}$,
Phase = 0.

b)



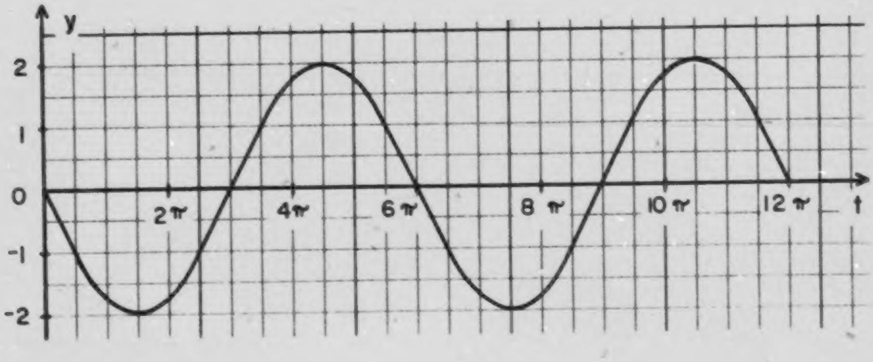
Amplitude = 2, Period = $\frac{4\pi}{3}$, Phase = 0 .

c)



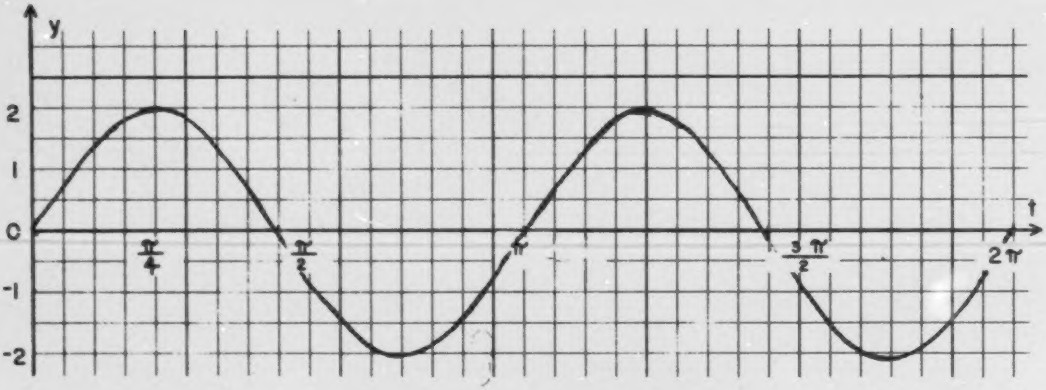
Amplitude = 3, Period = π, Phase = 0.

d)



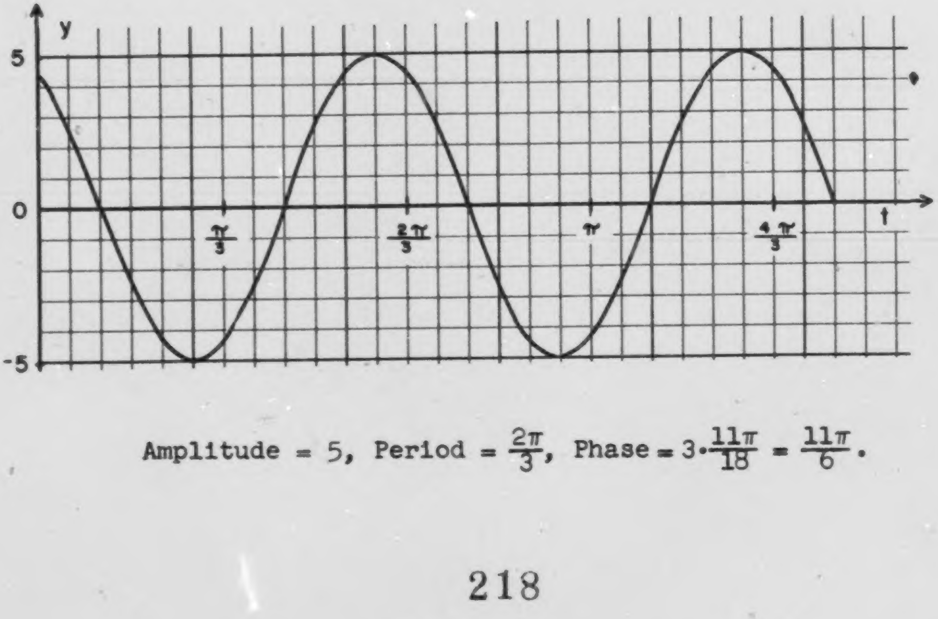
Amplitude = 2, Period = 6π, Phase = $\frac{1}{3}\cdot\frac{9\pi}{2} = \frac{3\pi}{2}$.

e)



Amplitude = 2, Period = π, Phase = $2\cdot\frac{3\pi}{4} = \frac{3\pi}{2}$.

f)



Amplitude = 5, Period = $\frac{2\pi}{3}$, Phase = $3\cdot\frac{11\pi}{18} = \frac{11\pi}{6}$.

3. a) $y = A\cos(\omega t - \alpha) = A\cos\omega t\cos\alpha + A\sin\omega t\sin\alpha$

$y = 4\sin\pi t - 3\cos\pi t$

$A\sin\alpha = 4, \quad A\cos\alpha = -3$

$A^2(\sin^2\alpha + \cos^2\alpha) = 16 + 9$

$A^2 = 25, \quad A = 5$

$\sin\alpha = \frac{4}{5}, \quad \cos\alpha = -\frac{3}{5},$

$\alpha \approx \pi - 0.927 \approx 2.215$

Answer: $y = 5\cos(\pi t - 2.215)$

b) $y = -4\sin\pi t + 3\cos\pi t$

$A\sin\alpha = -4, \quad A\cos\alpha = 3, \quad A = 5.$

$\sin\alpha = -\frac{4}{5}, \quad \cos\alpha = \frac{3}{5},$

$\alpha \approx 2\pi - 0.927 \approx 5.357$

Answer: $y = 5\cos(\pi t - 5.357)$

c) $y = -4\sin\pi t - 3\cos\pi t$

$A = 5, \quad \sin\alpha = -\frac{4}{5}, \quad \cos\alpha = -\frac{3}{5}.$

$\alpha \approx \pi + 0.927 \approx 4.069$

Answer: $y = 5\cos(\pi t - 4.069)$

d) $y = 3\sin\pi t + 4\cos\pi t$

$A = 5, \quad \sin\alpha = \frac{3}{5}, \quad \cos\alpha = \frac{4}{5}, \quad \alpha \approx 0.644$

Answer: $y = 5\cos(\pi t - 0.644)$

e) $y = 3\sin\pi t - 4\cos\pi t$

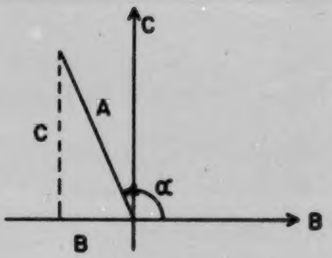
$A = 5, \quad \sin\alpha = \frac{3}{5}, \quad \cos\alpha = -\frac{4}{5}$

$\alpha \approx \pi - 0.644 \approx 2.498$

Answer: $y = 5\cos(\pi t - 2.498)$.

4. $A^2 = B^2 + C^2$

$\sin\alpha = \frac{C}{A}$, $\cos\alpha = \frac{B}{A}$.



Although the directions in this problem do not ask for the values of t at which the maxima and minima occur, they have been included in these solutions in case the question arises.

a) $A = 5$, $\sin\alpha = \frac{3}{5}$, $\cos\alpha = \frac{4}{5}$, $\alpha \approx 0.644$.

Hence, $3 \sin 2t + 4 \cos 2t \approx 5 \cos (2t - 0.644)$.

Maximum value, 5, occurs when $\cos (2t - 0.644) = 1$, or

$$2t - 0.644 = 0, \quad t = 0.322.$$

Minimum value, -5, occurs when $\cos (2t - 0.644) = -1$, or

$$2t - 0.644 = \pi, \quad t \approx 1.893.$$

The period $= \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$.

Hence, maximum values occur at $t \approx 0.322 + n\pi$ and minimum values at $t \approx 1.893 + n\pi$.

b) $A = \sqrt{4 + 9} = \sqrt{13}$, $\sin\alpha = \frac{2}{\sqrt{13}}$, $\cos\alpha = \frac{-3}{\sqrt{13}}$,

$\alpha \approx \pi - 0.589 \approx 2.553$.

Hence, $2 \sin 3t - 3 \cos 3t \approx \sqrt{13} \cos (3t - 2.553)$.

The period $= \frac{2\pi}{3}$. Maximum values, $\sqrt{13}$, occur when

$$3t - 2.553 = 0 + 2n\pi, \quad t \approx 0.851 + \frac{2n\pi}{3}.$$

Minimum values, $-\sqrt{13}$, occur when

$$3t - 2.553 = \pi + 2n\pi, \quad t \approx 1.898 + \frac{2n\pi}{3}.$$

4. c) $A = \sqrt{1 + 1} = \sqrt{2}$, $\sin\alpha = -\frac{1}{\sqrt{2}}$, $\cos\alpha = \frac{1}{\sqrt{2}}$, $\alpha = \frac{7\pi}{4}$

Hence, $-\sin(t/2) + \cos(t/2) = \sqrt{2}\cos(t/2 - 7\pi/4)$.

The period $= \frac{2\pi}{\frac{1}{2}} = 4\pi$. Maximum values, $\sqrt{2}$, occur when

$$\frac{t}{2} - \frac{7\pi}{4} = 0 + 2n\pi, \quad t = \frac{7\pi}{2} + 4n\pi.$$

Minimum values, $-\sqrt{2}$ occur when

$$\frac{t}{2} - \frac{7\pi}{4} = \pi + 2n\pi, \quad t = \frac{3\pi}{2} + 4n\pi.$$

5. $A\cos(\omega t - \alpha) + B\cos(\omega t - \beta)$

$$= A\cos\omega t\cos\alpha + A\sin\omega t\sin\alpha + B\cos\omega t\cos\beta + B\sin\omega t\sin\beta$$

$$= (A\cos\alpha + B\cos\beta)\cos\omega t + (A\sin\alpha + B\sin\beta)\sin\omega t$$

$= C\cos(\omega t - \gamma)$ when

$$C = \sqrt{(A\sin\alpha + B\sin\beta)^2 + (A\cos\alpha + B\cos\beta)^2}\ ,$$

$$\sin\gamma = \frac{A\sin\alpha + B\sin\beta}{C}, \text{ and } \cos\gamma = \frac{A\cos\alpha + B\cos\beta}{C}$$

Since A, B, α, and β are real numbers, it follows that C is a real number, and it is easy to show that

$$0 \le \sin^2\gamma \le 1,\ 0 \le \cos^2\gamma \le 1,\ \sin^2\gamma + \cos^2\gamma = 1,$$

and therefore γ is a real number.

6. a) From the solution in the text,

$$t = \frac{0.927}{\pi} \pm \frac{1}{3} + 2n.$$

So $t \approx 0.295 \pm 0.333 + 2n.$

The smallest positive value of t is

$$t \approx 0.295 + 0.333 = 0.628.$$

b) $3\cos\pi t + 4\sin\pi t = 5$

$5\cos(\pi t - 0.927) = 5$

$\cos(\pi t - 0.927) = 1$

This is satisfied when the argument of the cosine is

$$0 + 2n\pi.$$

Therefore,

$$\pi t - 0.927 = + 2n\pi,$$

or

$$t \approx \frac{0.927}{\pi} + 2n \approx 0.295 + 2n.$$

The smallest positive value of t is

$$t \approx 0.295.$$

c) $\sin 2t - \cos 2t = 1$

$\sqrt{2} \cos \left(2t - \frac{3\pi}{4}\right) = 1$

$\cos \left(2t - \frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

This is satisfied when the argument of the cosine is

$$\pm \frac{\pi}{4} + 2n\pi.$$

Therefore,

$$2t - \frac{3\pi}{4} = \pm \frac{\pi}{4} + 2n\pi,$$

or

$$t = \frac{3\pi}{8} \pm \frac{\pi}{8} + n\pi .$$

The smallest positive value of t is

$$t = \frac{\pi}{4} .$$

d) $4 \cos \pi t - 3 \sin \pi t = 0$

$5 \cos (\pi t - 5.640) = 0$

$\cos (\pi t - 5.640) = 0.$

This is satisfied when the argument of the cosine is

$$\pm \frac{\pi}{2} + 2n\pi.$$

Therefore,

$$\pi t - 5.640 = \pm \frac{\pi}{2} + 2n\pi,$$

or

$$t \approx \frac{5.640}{\pi} \pm \frac{1}{2} + 2n, \approx 1.795 \pm 0.5 + 2n.$$

The smallest positive value of t is

$$t \approx 1.795 + 0.5 - 2 \approx 0.295.$$

e) $4 \cos \pi t + 3\ \pi t = 1$

$5 \cos (\pi t - 0.644) = 1$

$\cos (\pi t - 0.644) = 0.2$

This is satisfied when the argument of the cosine is approximately $\pm 1.369 + 2n\pi$ (from Table I). Therefore,

$$\pi t - 0.644 \approx \pm 1.369 + 2n\pi,$$

or

$$t \approx \frac{0.644 \pm 1.369}{\pi} + 2n.$$

The smallest positive value of t is

$$t \approx \frac{0.644 + 1.369}{\pi} + 0 \approx 0.641.$$

7. Given $y = B \cos (\mu t - \beta)$.
We may clearly assume that $0 < \beta < 2\pi$.

1. If μ and B are positive, we set $\mu = \omega$, $B = A, \beta = \alpha$
2. If μ is positive and B is negative, set $\mu = \omega$, $B = -A$.
   Then $y = A\,[-\cos (\omega t - \beta)] = A \cos (\omega t - \beta \pm \pi)$.
   If $0 \leq \beta < \pi$, take $\alpha = \beta + \pi$.
   If $\pi \leq \beta < 2\pi$, take $\alpha = \beta - \pi$.
3. If μ is negative, set $\mu = -\omega$.
   Then $y = B \cos (-\omega t - \beta) = B \cos (\omega t + \beta)$
   $= B \cos [\omega t - (2\pi - \beta)]$
   $= B \cos (\omega t - \beta')$.

Proceed as in 1 and 2.

5-9. Identities.

The identities dealt with here are somewhat more difficult than earlier ones. It may be necessary for you to work a few additional examples on the blackboard to help get the students started.

Answers to Exercises 5-9

1. $\cos(x + y) = \cos x \cos y - \sin x \sin y$

$\cos(x - y) = \cos x \cos y + \sin x \sin y$

$\cos(x + y) + \cos(x - y) = 2 \cos x \cos y$

$$\cos x \cos y = \tfrac{1}{2}[\cos(x + y) + \cos(x - y)] \qquad (1)$$

$\cos(x - y) - \cos(x + y) = 2 \sin x \sin y$

$$\sin x \sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)] \qquad (2)$$

$\sin(x + y) = \sin x \cos y + \cos x \sin y$

$\sin(x - y) = \sin x \cos y - \cos x \sin y$

$\sin(x + y) + \sin(x - y) = 2 \sin x \cos y$

$$\sin x \cos y = \tfrac{1}{2}[\sin(x + y) + \sin(x - y)] \qquad (3)$$

2. $\cos m\alpha \cdot \cos n\alpha = \tfrac{1}{2}[\cos(m + n)\alpha + \cos(m - n)\alpha]$

$\sin m\alpha \cdot \sin n\alpha = \tfrac{1}{2}[\cos(m - n)\alpha - \cos(m + n)\alpha]$

$\sin m\alpha \cdot \cos n\alpha = \tfrac{1}{2}[\sin(m + n)\alpha + \sin(m - n)\alpha]$

3. $$\sin\alpha + \sin\beta = 2 \sin\frac{\alpha + \beta}{2} \cos\frac{\alpha - \beta}{2} \qquad (6)$$

Replace β by $-\beta$ and $\sin\beta$ by $-\sin\beta$. Then

$\sin\alpha - \sin\beta = 2 \sin\frac{\alpha - \beta}{2} \cos\frac{\alpha + \beta}{2}$.

4. In (5), let $\alpha = x$ and $\beta = \frac{\pi}{2} - x$. Then

$$\begin{aligned}\cos\alpha - \cos\beta &= \cos x - \sin x \\ &= -2 \sin\frac{\pi}{4} \sin\left(\frac{2x - \pi/2}{2}\right) \\ &= -\sqrt{2} \sin\left(x - \frac{\pi}{4}\right) \\ &= \sqrt{2} \sin\left(\frac{\pi}{4} - x\right).\end{aligned}$$

5. a) $\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6}$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6} - \sqrt{2}}{4}$$

5. b) $\cos \frac{5\pi}{12} = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6} - \sqrt{2}}{4}.$$

or

$$\cos\frac{5\pi}{12} = \sin\left(\frac{\pi}{2} - \frac{5\pi}{12}\right) = \sin\frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}.$$

c) $\tan\frac{7\pi}{12} = \tan\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \frac{\tan \pi/4 + \tan \pi/3}{1 - \tan \pi/4 \tan \pi/3}$

$$= \frac{1 + \sqrt{3}}{1 - \sqrt{3}} = -(2 + \sqrt{3})$$

d) $\cos\frac{11\pi}{12} = \cos\left(\frac{3\pi}{4} + \frac{\pi}{6}\right)$

$$= \cos\frac{3\pi}{4}\cos\frac{\pi}{6} - \sin\frac{3\pi}{4}\sin\frac{\pi}{6}$$

$$= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{-(\sqrt{6} + \sqrt{2})}{4}.$$

6. a) $\cos^4\theta - \sin^4\theta = (\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$

$$= \cos^2\theta - \sin^2\theta$$

$$= \cos 2\theta$$

b) $\cos^2\alpha - \sin^2\alpha = (1 - \sin^2\alpha) - \sin^2\alpha$

$$= 1 - 2\sin^2\alpha$$

c) $\frac{1}{\cos^2\alpha} - 1 = \frac{1 - \cos^2\alpha}{\cos^2\alpha}$

$$= \frac{\sin^2\alpha}{\cos^2\alpha}$$

$$= \tan^2\alpha$$

Neither side is defined if $\cos\alpha = 0$, that is, if

$$\alpha = \pm\frac{\pi}{2} + m\pi.$$

d) $\cos(\alpha - \pi) = \cos\alpha\cos\pi + \sin\alpha\sin\pi = -\cos\alpha$

$\cos(\alpha + \pi) = \cos\alpha\cos\pi - \sin\alpha\sin\pi = -\cos\alpha$

Hence, $\cos(\alpha - \pi) = \cos(\alpha + \pi)$.

e) $\tan \frac{1}{2}\theta = \dfrac{\sin \frac{1}{2}\theta}{\cos \frac{1}{2}\theta} = \dfrac{2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta}{2 \cos^2 \frac{1}{2}\theta}$

$$= \frac{\sin \theta}{1 + \cos \theta}$$

f) $\cos^2 \frac{1}{2}\theta = \frac{1}{2}(1 + \cos \theta)$

$$= \frac{\tan \theta + \cos \theta \cdot \tan \theta}{2 \tan \theta}$$

$$= \frac{\tan \theta + \sin \theta}{2 \tan \theta} \qquad (\tan \theta \neq 0)$$

g) $(\sin \frac{1}{2}\alpha + \cos \frac{1}{2}\alpha)^2$

$$= \sin^2 \tfrac{1}{2}\alpha + 2 \sin \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha + \cos^2 \tfrac{1}{2}\alpha$$

$$= 1 + 2 \sin \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha$$

$$= 1 + \sin (2 \cdot \tfrac{1}{2}\alpha)$$

$$= 1 + \sin \alpha$$

h) $(\sin \theta + \cos \theta)^2 = \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta$

$$= 1 + 2 \sin \theta \cos \theta$$

$$= 1 + \sin 2\theta$$

i) $\dfrac{2 \tan \theta}{1 + \tan^2 \theta} = \dfrac{\dfrac{2 \sin \theta}{\cos \theta}}{1 + \dfrac{\sin^2 \theta}{\cos^2 \theta}} = \dfrac{2 \sin \theta \cos \theta}{\cos^2 \theta + \sin^2 \theta}$

$$= 2 \sin \theta \cos \theta = \sin 2\theta$$

j) $\dfrac{1 + \cos \theta}{\sin \theta} + \dfrac{\sin \theta}{1 + \cos \theta} = \dfrac{(1 + \cos \theta)^2 + \sin^2 \theta}{\sin \theta (1 + \cos \theta)}$

$$= \frac{1 + 2 \cos \theta + \cos^2 \theta + \sin^2 \theta}{\sin \theta (1 + \cos \theta)}$$

$$= \frac{2(1 + \cos \theta)}{\sin \theta (1 + \cos \theta)}$$

$$= \frac{2}{\sin \theta}.$$

[Neither side of the identity is defined if $\theta = n\pi$.]

k) $$\frac{\sin 2\alpha}{\sin \alpha} - \frac{\cos 2\alpha}{\cos \alpha} = \frac{\sin 2\alpha \cos \alpha - \cos 2\alpha \sin \alpha}{\sin \alpha \cos \alpha}$$

$$= \frac{\sin(2\alpha - \alpha)}{\sin \alpha \cos \alpha} = \frac{\sin \alpha}{\sin \alpha \cos \alpha}$$

$$= \frac{1}{\cos \alpha}.$$

Alternatively,

$$\frac{\sin 2\alpha}{\sin \alpha} - \frac{\cos 2\alpha}{\cos \alpha} = \frac{2 \sin \alpha \cos \alpha}{\sin \alpha} - \frac{\cos^2 \alpha - \sin^2 \alpha}{\cos \alpha}$$

$$= 2 \cos \alpha - \frac{\cos^2 \alpha - \sin^2 \alpha}{\cos \alpha}$$

$$= \frac{2 \cos^2 \alpha - \cos^2 \alpha + \sin^2 \alpha}{\cos \alpha}$$

$$= \frac{\cos^2 \alpha + \sin^2 \alpha}{\cos \alpha}$$

$$= \frac{1}{\cos \alpha}.$$

The left side is undefined if $\sin \alpha \cos \alpha = 0$, hence if $\sin 2\alpha = 0$, that is, if $\alpha = n\frac{\pi}{2}$.

7. a) $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

b) $\cos^4 x = \frac{1}{4}(1 + 2 \cos 2x + \cos^2 2x)$

$= \frac{1}{4}[1 + 2 \cos 2x + \frac{1}{2}(1 + \cos 4x)]$

$= \frac{1}{8}[3 + 4 \cos 2x + \cos 4x]$

8. $\cos 2x = 1 - 2 \sin^2 x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\sin^4 x = \frac{1}{4}(1 - 2 \cos 2x + \cos^2 2x)$

$= \frac{1}{4}[1 - 2 \cos 2x + \frac{1}{2}(1 + \cos 4x)]$

$= \frac{1}{8}[3 - 4 \cos 2x + \cos 4x]$

9. a) $\sin 2\theta \cos\theta - \cos 2\theta \sin\theta = \sin(2\theta - \theta) = \sin\theta$

b) $\sin(x - y)\cos z + \sin(y - z)\cos x$

$= (\sin x \cos y - \cos x \sin y)\cos z$

$\quad + (\sin y \cos z - \cos y \sin z)\cos x$

$= \sin x \cos y \cos z - \cos y \sin z \cos x$

$= (\sin x \cos z - \sin z \cos x)\cos y$

$= \sin(x - z)\cos y$

c) $\sin 3x \sin 2x = \frac{1}{2}[\cos(3x - 2x) - \cos(3x + 2x)]$

$= \frac{1}{2}(\cos x - \cos 5x)$ [See (2)]

d) $\cos\theta - \sin\theta \tan 2\theta = \cos\theta - \sin\theta \dfrac{\sin 2\theta}{\cos 2\theta}$

$= \dfrac{\cos\theta \cos 2\theta - \sin\theta \sin 2\theta}{\cos 2\theta}$

$= \dfrac{\cos(2\theta + \theta)}{\cos 2\theta} = \dfrac{\cos 3\theta}{\cos 2\theta}$ (Valid if $\cos 2\theta \neq 0$.)

e) $\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$

$= 2\sin\theta \cos^2\theta + (1 - 2\sin^2\theta)\sin\theta$

$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

Hence $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$.

f) $\sin 3x + \sin x = 2\sin 2x \cos x$ [by (6)]

Hence, $\sin x + \sin 2x + \sin 3x = \sin 2x[1 + 2\cos x]$

g) $\dfrac{1 + \tan x}{1 - \tan x} = \dfrac{1 + \frac{\sin x}{\cos x}}{1 - \frac{\sin x}{\cos x}} = \dfrac{\cos x + \sin x}{\cos x - \sin x}$

Hence

$$\left(\frac{1 + \tan x}{1 - \tan x}\right)^2 = \frac{\cos^2 x + 2\sin x \cos x + \sin^2 x}{\cos^2 x - 2\sin x \cos x + \cos^2 x}$$

$$= \frac{1 + 2\sin x \cos x}{1 - 2\sin x \cos x}$$

$$= \frac{1 + \sin 2x}{1 - \sin 2x}$$

[sec. 5-9]

5-10 and 5-11. Tangents to Graphs of Sine and Cosine.

Following the general type of argument used in Chapters 3 and 4, we show that near the origin, the graph of $y = \sin x$ lies in the wedge between $y = x$ and $y = (1 - \epsilon)x$ for any positive ϵ, however small, and hence that the line tangent at the origin to $y = \sin x$ is $y = x$. A similar argument shows that the line tangent at $(0, 1)$ to $y = \cos x$ is $y = 1$.

We then generalize these results to an arbitrary point $(h, f(h))$ of the graph by expressing $\sin x$ and $\cos x$ in terms of $x - h$. This leads to the results

$$\sin' = \cos \quad \text{and} \quad \cos' = -\sin .$$

Answers to Exercises 5-11

1. a) Putting $h = \frac{\pi}{3}$ in (1), we have

$$y = \sin\frac{\pi}{3} + \left(\cos\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

$$= \frac{1}{2}x + \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)$$

b) $y = -\frac{1}{2}x + \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$

c) $y = .5403x + .3012$

2. a) Using (2),

$$y = \cos\frac{\pi}{6} - \left(\sin\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) = -\frac{1}{2}x + \left(\frac{\sqrt{3}}{2} + \frac{\pi}{12}\right)$$

b) $y = -.9086x + 1.3996$

3. a) $0 - \sin 0 = 0$

b) $.1 - \sin .1 = .1 - .0998 = .0002$

c) $.2 - .1987 = .0013$

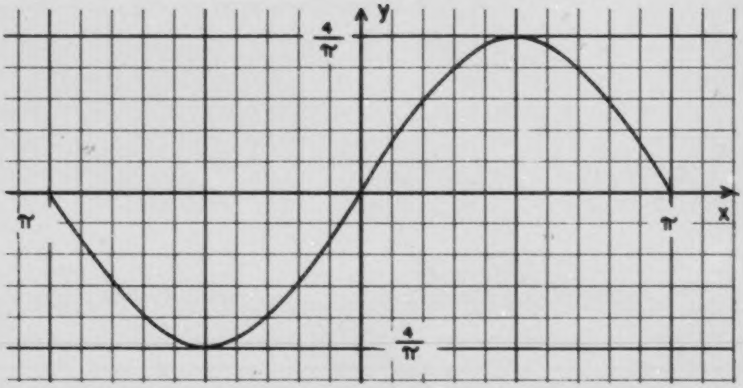
d) $.3 - .2955 = .0045$

5-12. Analysis of General Waves

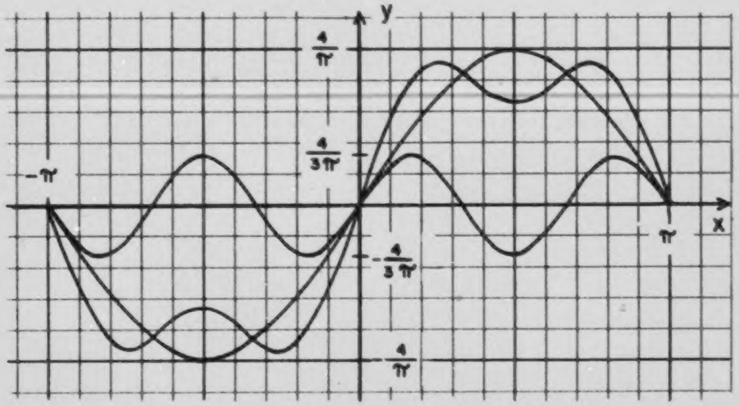
The purpose of this section is to provide the student with some idea of the power of simple circular functions, and to show how they can be used to approximate much more complex periodic functions. We do not intend that the student use Fourier's Theorem, but only that he understand what it says, and what it implies.

Answers to Exercises 5-12

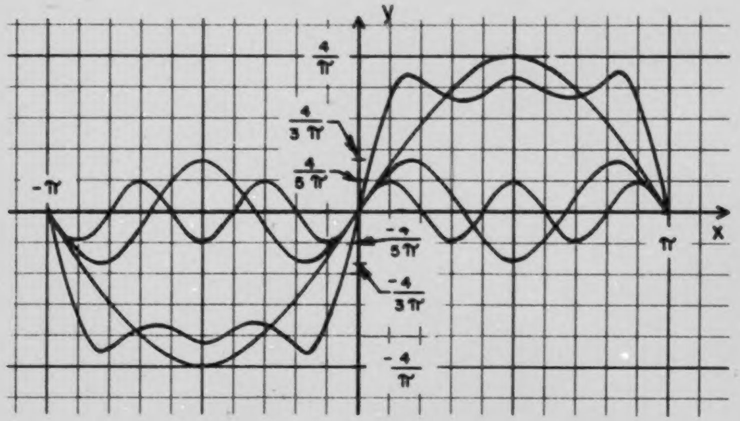
1. a)



b)



c)



2. a) $2\pi, \frac{2\pi}{3}, \frac{2\pi}{5}, \ldots$

   b) The cosine terms; also the terms $B_n \sin nx$, n even.

   (In our case, $a = 2\pi$.) The function being represented has the property that $f(-x) = -f(x)$ [odd function]. This property holds for $\sin nx$ but not $\cos nx$. Moreover, $f(x)$ has the property that, $f(\pi - x) = f(x)$. This property does not hold for $\sin 2kx$, k integral, since $\sin 2k(\pi - x) = -\sin 2kx$. It does hold for $\sin(2k + 1)x$.

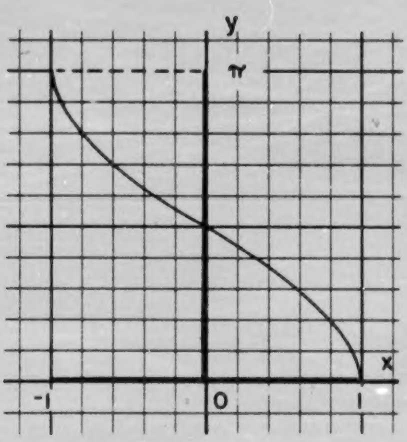
5-13. Inverse Circular Functions and Trigonometric Equations

If the student has not covered this material before, you will have to go very slowly, since there is some new notation involved as well as the idea of the necessity of restricting the domain to obtain a function with an inverse. You should make sure that the student understands why such restriction is necessary. It will probably be helpful to refer back to Section 4-8. The solution of trigonometric equations is presented briefly, but the methods used in the examples are quite general.

Answers to Exercises 5-13

1. $y = \cos^{-1}x$ Domain $\{x: -1 \leq x \leq 1\}$

Range $\{y: 0 \leq y \leq \pi\}$



2. a) $\sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$ c) $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

b) $\cos^{-1}(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$ d) $\cos^{-1} 1 - \sin^{-1}(-1) = \frac{\pi}{2}$

3. a) $\sin(\cos^{-1} .73) = 0.68$

b) $\cos(\sin^{-1}(-0.47)) = 0.88$

c) $\sin[\cos^{-1}\frac{3}{5} + \sin^{-1}(-\frac{3}{5})] = \sin(\cos^{-1}\frac{3}{5})\cos(\sin^{-1}(-\frac{3}{5}))$

$+ \cos(\cos^{-1}\frac{3}{5})\sin(\sin^{-1} - \frac{3}{5}) = \frac{4}{5}\cdot\frac{4}{5} + \frac{3}{5}(-\frac{3}{5}) = \frac{7}{25}$

d) $\sin[2\cos^{-1}\frac{5}{13}]$. Let $\cos^{-1}\frac{5}{13} = \theta$

$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{12}{13}\cdot\frac{5}{13} = \frac{120}{169}$.

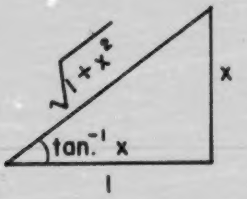
4. a) $\sin (\cos^{-1} \frac{2}{3}) = \cos (\sin^{-1} \frac{2}{3})$

$\cos^{-1} \frac{2}{3} = \frac{\pi}{2} - \sin^{-1} \frac{2}{3}$ Hence,

$\sin (\cos^{-1} \frac{2}{3}) = \sin (\frac{\pi}{2} - \sin^{-1} \frac{2}{3}) = \cos (\sin^{-1} \frac{2}{3})$.

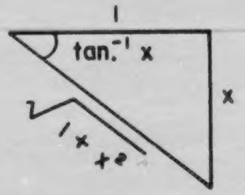
b) No. It is, however, true for $0 \leq x \leq 1$.

5. $\sin (\tan^{-1} x) = \frac{x}{\sqrt{1 + x^2}}$



$x > 0$

or



$x < 0$

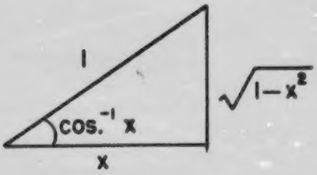
The result follows immediately from the figures.

6. a) $\sin (2 \tan^{-1} x) = 2 \sin (\tan^{-1} x) \cos (\tan^{-1} x)$

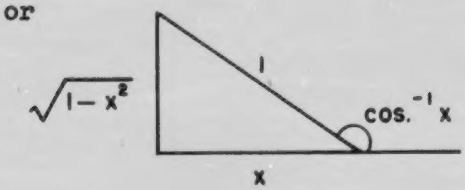
$$= 2 \frac{x}{\sqrt{1 + x^2}} \cdot \frac{1}{\sqrt{1 + x^2}} = \frac{2x}{1 + x^2}$$

b) $\tan (2 \tan^{-1} x) = \frac{2 \tan (\tan^{-1} x)}{1 - \tan^2 (\tan^{-1} x)} = \frac{2x}{1 - x^2}$

c) $\tan (\cos^{-1} x) = \frac{\sqrt{1 - x^2}}{x}$



or



d) $\sin(\sin^{-1} x + \cos^{-1} x) = \sin(\sin^{-1} x)\cos(\cos^{-1} x)$
$+ \cos(\sin^{-1} x)\sin(\cos^{-1} x)$

$$= x \cdot x + \sqrt{1 - x^2} \cdot \sqrt{1 - x^2}$$
$$= x^2 + 1 - x^2 = 1$$

or $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ and $\sin \frac{\pi}{2} = 1$.

Valid only if $0 \leq x \leq 1$.

7. a) $\sin x + \cos x = 0$, $\tan x = -1$

$x = -\frac{\pi}{4} + n\pi$

b) $4\cos^2 x - 1 = 0$, $\cos x = \pm\frac{1}{2}$

$x = \pm\frac{\pi}{3} + 2n\pi$ or $\pm\frac{2\pi}{3} + 2n\pi$

c) $3\tan x - \sqrt{3} = 0$, $\tan x = \frac{\sqrt{3}}{3}$

$x = \frac{\pi}{6} + n\pi$

d) $4\tan x + \sin 2x = 0$

$4\frac{\sin x}{\cos x} + 2\sin x\cos x = 0$ $(x \neq \frac{\pi}{2} + n\pi)$

$\sin x(4 + 2\cos^2 x) = 0$

$\sin x = 0$, $x = n\pi$.

8. a) $2\cos x - \sin x = 1$

$2\cos x - 1 = \pm\sqrt{1 - \cos^2 x}$

$4\cos^2 x - 4\cos x + 1 = 1 - \cos^2 x$

$5\cos^2 x - 4\cos x = 0$

$\cos x(5\cos x - 4) = 0$ $\cos x = 0$ or $\cos x = \frac{4}{5}$.

If $\cos x = 0$, $\sin x = -1$ to satisfy equation.

Then $x = \frac{3\pi}{2} + 2n\pi$.

If $\cos x = \frac{4}{5}$, $\sin x = \frac{3}{5}$, $x = .64 + 2n\pi$.

b) $9 \cos^2 x + 6 \cos x - 8 = 0$

$(3 \cos x - 2)(3 \cos x + 4) = 0$

$\cos x = \frac{2}{3}$. Since $\cos x = -\frac{4}{3}$ is impossible,

$x = \pm .84 + 2n\pi$.

c) $\tan x = \frac{1}{\tan x}$

$\tan^2 x - 1 = 0$

$\tan x = 1$ $\qquad$ $\tan x = -1$

$x = \frac{\pi}{4} + n\pi$ $\qquad$ $x = \frac{3\pi}{4} + n\pi$

d) $\cos 2x - 1 = \sin x$

$1 - 2 \sin^2 x - 1 = \sin x$

$-2 \sin^2 x = \sin x$

$\sin x = 0, \quad \sin x = -\frac{1}{2}$

$x = n\pi, \qquad x = -\frac{\pi}{6} + 2n\pi, \quad -\frac{5\pi}{6} + 2n\pi$.

9. a) $2 \sin^{-1} x = \frac{\pi}{4} \quad \sin^{-1} x = \frac{\pi}{8}$

$x = \sin \frac{\pi}{8} = 0.38$.

b) $\sin 2x = \cos (\pi - x) \qquad \cos (\pi - x) = -\cos x$

$2 \sin x \cos x = -\cos x$

$\sin x = -\frac{1}{2} \qquad \cos x = 0$

$x = -\frac{\pi}{6} + 2n\pi, \ -\frac{5\pi}{6} + 2n\pi. \qquad x = \frac{\pi}{2} + n\pi$

c) $2 \sin^{-1} 2x = 3$

$\sin^{-1} 2x = \frac{3}{2}$

$2x = 1.00^-$

$x = .49^+$

d) $3 \sin 2x = 2$ $\qquad \sin 2x = \frac{2}{3}$

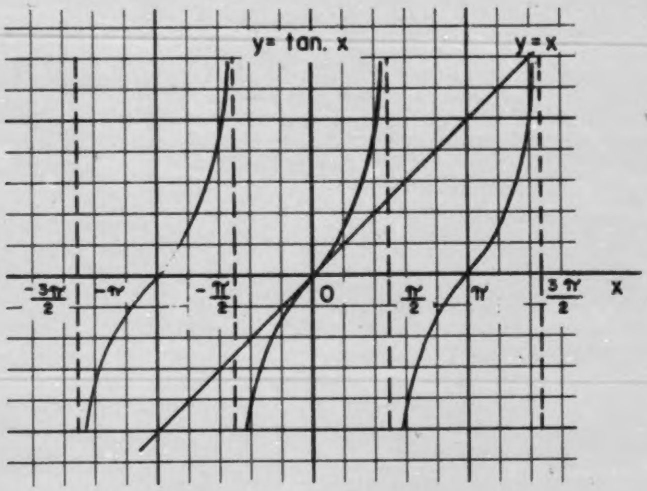
$2x = .73 + 2n\pi, \quad 2.41 + 2n\pi$

$x = .37 + n\pi, \quad 1.21 + n\pi$

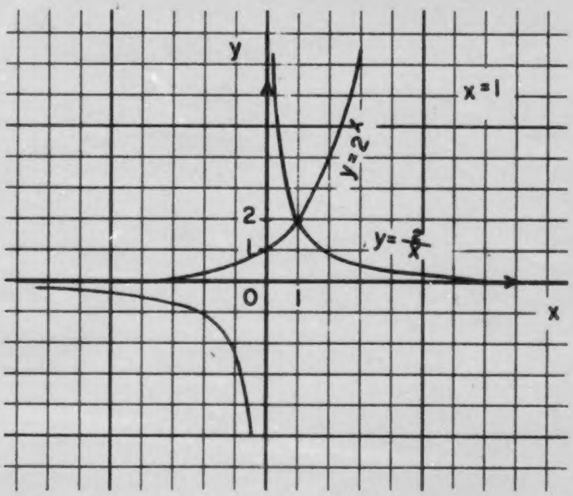
10. a) $x = \tan x$

$x = 0, \pm 4.49$

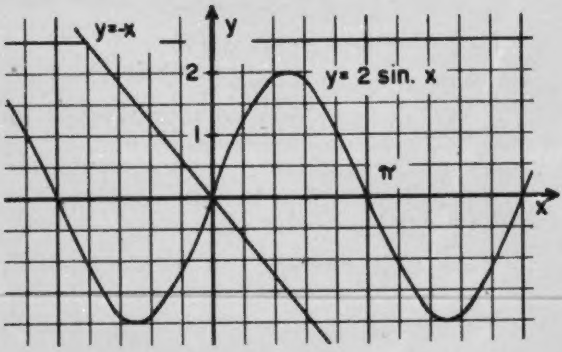
and other solutions corresponding to other intersections of graphs shown.



b) $x \cdot 2^x = 2$ $\qquad 2^x = \frac{2}{x}$

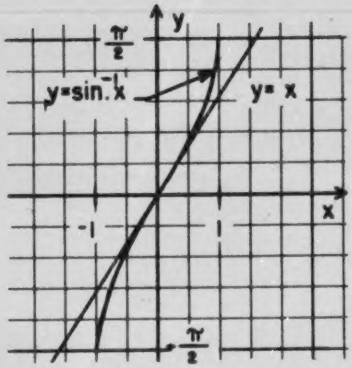


c) $x + 2 \sin x = 0$ $\qquad\qquad$ $-x = 2 \sin x$

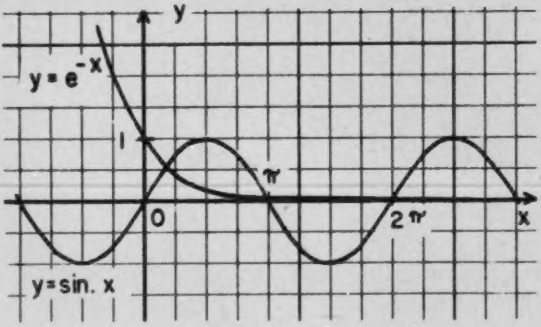


$x = 0$ is the only solution.

d) $x = \sin^{-1} x$ $\qquad\qquad$ $x = 0$ is the only solution.



e) $\sin x = e^{-x}$ $\qquad$ $x \approx .59$, 3.12, and so on.



Miscellaneous Exercises

1. a) $|\sin(\pi + x)| = |-\sin x| = |\sin x|$. Periodic with period π.

b) Since $[x + 1] = [x] + 1$, we have
$f(x + 1) = (x + 1) - [x + 1] = x + 1 - [x] - 1$
$= x - [x] = f(x)$. periodic with period 1.

c) $y = x \sin x$. $\qquad$ not periodic

d) $y = \sin^2 x$, $\sin^2(\pi + x) = [-\sin x]^2 = \sin^2 x$.
period π.

e) $y = \sin x^2$ $\qquad$ not periodic

f) $y = \dfrac{\sin x + 2\cos x}{2\sin x + \cos x}$. This is not defined if

$\sin x = \dfrac{-\cos x}{2}$, or $\tan x = -\dfrac{1}{2}$, and therefore at

$x = \tan^{-1}(-\frac{1}{2}) + n\pi$. Defined for all other values of x.

$$f(x + \pi) = \frac{\sin (x + \pi) + 2 \cos (x + \pi)}{2 \sin (x + \pi) + \cos (x + \pi)} = \frac{- \sin x - 2 \cos x}{- 2 \sin x - \cos x}$$

$$= \frac{\sin x + 2 \cos x}{2 \sin x + \cos x} = f(x).$$ Therefore periodic with period π.

g) $y = \sin x + |\sin x|$
$\sin x$ is periodic with period 2π
$|\sin x|$ is periodic with period π
$\therefore$ The sum is periodic with period the lcm of 2π and π, or 2π.

h) $y = \sin x + \sin (\sqrt{2}\, x)$
$\sin x$ is periodic with period 2π
$\sin \sqrt{2}\, x$ is periodic with period $\frac{2\pi}{\sqrt{2}}$

But there is no lcm of two incommensurable numbers, and therefore no period for the function.

2. The decimal expansion of $\frac{110}{909}$ is $.\dot{1}21\dot{0}$; hence the range is $\{0, 1, 2\}$

$\therefore f(x + 4) = f(x)$ $\qquad$ $\therefore$ periodic with period 4.
$f(97) = f(1 + 4 \cdot 24) = f(1) = 1.$

3. $f(x + 2) = f(x)$ $\qquad$ $f(-x) = - f(x)$ $\qquad$ $f(\frac{1}{2}) = 3$

a) $f(\frac{9}{2}) = f(\frac{1}{2} + 2 \cdot 2) = f(\frac{1}{2}) = 3$

b) $f(\frac{7}{2}) = f(- \frac{1}{2} + 4) = f(- \frac{1}{2}) = - f(\frac{1}{2}) = - 3$

c) $f(9) + f(-7) = f(1 + 4 \cdot 2) + f(-1 + 4 \cdot 2) = f(1) + f(-1)$
$= f(1) - f(1) = 0$

4. a) $\frac{22}{7}$ radians → $(\frac{22}{7} \cdot \frac{180}{\pi})^{\circ} \approx 180^{\circ}$

b) $\frac{2}{\pi}$ radians → $(\frac{2}{\pi} \cdot \frac{180}{\pi})^{\circ} = (\frac{360}{\pi^2})^{\circ}$

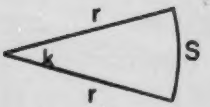
5. a) $87^\circ \rightarrow \frac{87\pi}{180}$ radians

   b) $\left(\frac{2}{\pi}\right)^\circ \rightarrow \frac{2}{\pi}\cdot\frac{\pi}{180} = \frac{1}{90}$ radians

   c) $\left(\frac{\pi}{2}\right)^\circ \rightarrow \frac{\pi}{2}\cdot\frac{\pi}{180} = \frac{\pi^2}{360}$ radians

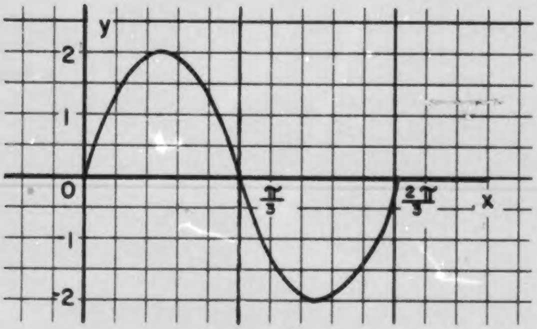
6. $S = kr$

   $c = 2r + kr$ or $r = \frac{c}{2 + k}$



   $$\text{Area} = \frac{k}{2}\cdot r^2 = \frac{k}{2}\cdot\frac{c^2}{(2 + k)^2} = \frac{kc^2}{2(k + 2)^2}$$

7. a) $y = 2 \sin 3x$: period $\frac{2\pi}{3}$, amplitude 2, range $[-2, 2]$.

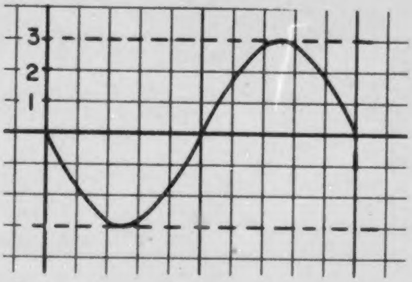


   b) $y = -3 \sin 2\pi x$

   period $\frac{2\pi}{2\pi} = 1$

   Amplitude + 3

   Range $[-3, 3]$

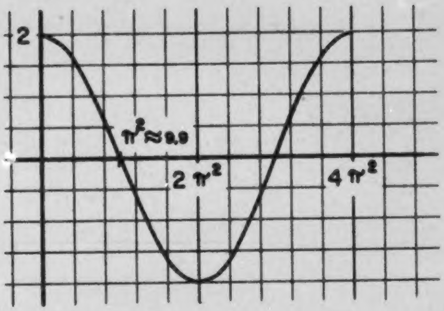


c) $y = 2 \cos \frac{x}{2\pi}$

period $\frac{2\pi}{\frac{1}{2\pi}} = 4\pi^2$

Amplitude 2

Range [- 2, 2]



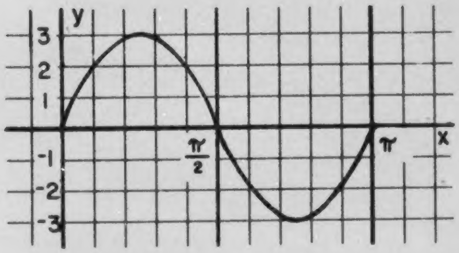
d) $y = 6 \sin x \cos x$. Transform the function first.

$y = 3 \sin 2x$.

period $\frac{2\pi}{2} = \pi$

Amplitude 3

Range [- 3, 3]



e) $y = \sqrt{3} \sin 2x + \cos 2x$. Transform the function first

$y = 2(\frac{\sqrt{3}}{2} \sin 2x + \frac{1}{2} \cos 2x)$ but $\sin \frac{\pi}{6} = \frac{1}{2}$ $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

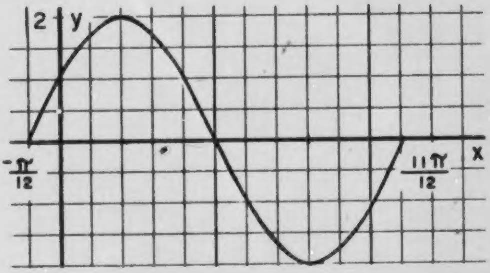
$y = 2(\sin 2x \cos \frac{\pi}{6} + \cos 2x \sin \frac{\pi}{6})$

$y = 2 \sin (2x + \frac{\pi}{6})$

period = $\frac{2\pi}{2} = \pi$

Amplitude = 2

Range [- 2, 2]



8. $f: x \rightarrow 3\cos\left(2\pi x + \frac{\pi}{2}\right) = -3\sin 2\pi x$

$$= 3\sin(2\pi x + \pi)$$

We can use $A = 3,\ B = 2\pi,\ C = \pi$

or $A = -3,\ B = 2\pi,\ C = 0$

9. $Q = Q_o \sin\left[\frac{t}{\sqrt{LC}} + \frac{\pi}{2}\right]$ $L = 0.4$ $C = 10^{-5}$

$$\frac{t}{\sqrt{LC}} = \frac{t}{\sqrt{(4)(10^{-6})}} = \frac{t}{2\cdot 10^{-3}} = 500t$$

$$Q = Q_o \sin\left(500t + \frac{\pi}{2}\right) = Q_o \cos 500t$$

a) period $= \frac{2\pi}{500}$, frequency $= \frac{250}{\pi}$

b) $Q = 0$ if $\cos\ 500t = 0$

This is zero for the first time at $t = \frac{1}{4}\cdot\frac{\pi}{250} = \frac{\pi}{1000}$

c) $Q = .5Q_o$ if $\cos 500t = .5 = \frac{1}{2}$; $500t = \frac{\pi}{3}$ or $t = \frac{\pi}{1500}$

d) $Q = .5Q_o$ for second time $500t = \frac{5\pi}{3}$ or $t = \frac{\pi}{300}$

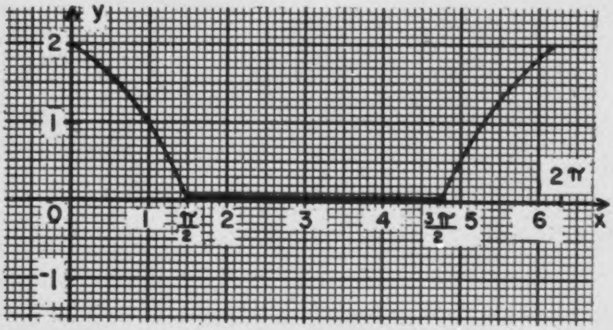
10. $\tan\frac{3x}{2} = \frac{\cos x - \cos 2x}{\sin 2x - \sin x} = -\frac{\cos 2x - \cos x}{\sin 2x - \sin x}$

$$= -\frac{-2\sin\frac{1}{2}(2x + x)\sin\frac{1}{2}(2x - x)}{2\cos\frac{1}{2}(2x + x)\sin\frac{1}{2}(2x - x)}$$

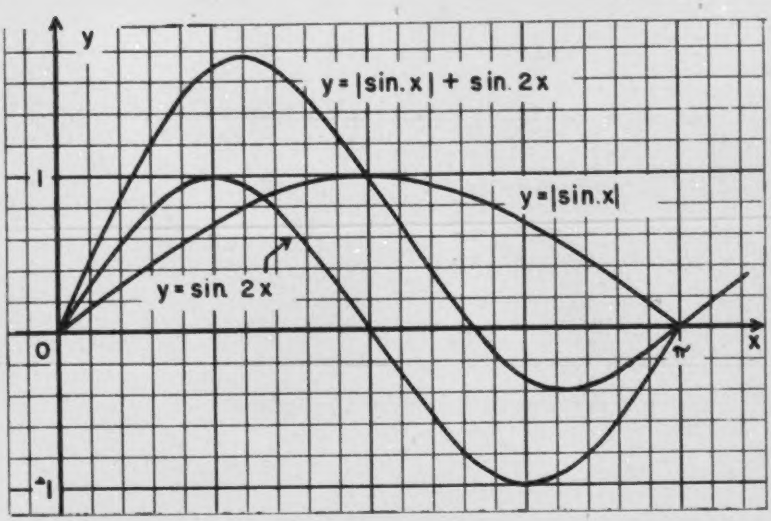
$$= \frac{\sin\frac{3x}{2}}{\cos\frac{3x}{2}} \qquad = \tan\frac{3x}{2}\ .$$

11. Sketch graph of:

a) $y = \cos x + |\cos x|$



b) $y = |\sin x| + \sin 2x$.



*12. Prove:

a) $|\sin x + \cos x| \le \sqrt{2}$

$$|\sin x + \cos x| = \left| \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right] \right|$$

$$= \sqrt{2}\,\left|\sin\left(x + \frac{\pi}{4}\right)\right|$$

and $|\sin \theta| \le 1$. q.e.d.

b) $$|\sqrt{3}\sin x + \cos x| = \left| 2\left[\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x\right] \right|$$

$$= |2|\,\left|\sin\left(x + \frac{\pi}{6}\right)\right|$$

≤ 2 since $|\sin \theta| \le 1$.

*13. a) $f(x) = 2x + 3 \rightarrow x = \dfrac{f(x) - 3}{2}$, $y = \dfrac{f(y) - 3}{2}$

$$x + y = \frac{f(x) + f(y) - 6}{2}$$

$$f(x + y) = 2(x + y) + 3 = 2\cdot\frac{f(x) + f(y) - 6}{2} + 3$$

$$= f(x) + f(y) - 3$$

b) $$f(x + y) = \frac{11f(x) + 11f(y) - 13f(x)f(y) - 12}{3f(x) + 3f(y) - 4f(x)f(y) + 4}.$$

c) $f(x + y) = \frac{1}{2}f(x)f(y)$

*14. $\sin(\cos x) = \cos(\sin x)$ has no solution.

Proof: For any $y \in R$, $\cos y = \sin(\frac{\pi}{2} - y)$. Put $y = \sin x$;

then $$\cos(\sin x) = \sin(\tfrac{\pi}{2} - \sin x)$$

and the given equation then becomes

$$\sin(\cos x) = \sin(\tfrac{\pi}{2} - \sin x)$$

Hence either $$\cos x = \tfrac{\pi}{2} - \sin x + 2n\pi \qquad (a)$$

or $$\cos x = \tfrac{\pi}{2} + \sin x + 2n\pi \qquad (b)$$

From (a), $\sin x + \cos x = \frac{\pi}{2} + 2n\pi = \dfrac{(4n + 1)\pi}{2}$

$$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{(4n + 1)\pi}{2\sqrt{2}}$$

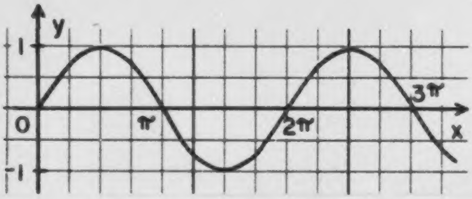
$$\sin(x + \tfrac{\pi}{4}) = \frac{(4n + 1)\pi}{2\sqrt{2}} \approx 1.11\,(4n + 1),$$

which is impossible because there is no integer n such that

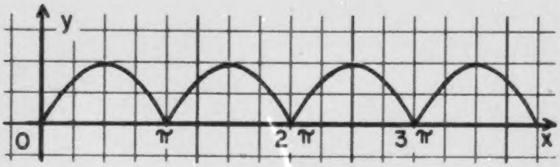
$$-1 \leq \frac{(4n + 1)\pi}{2\sqrt{2}} \leq 1.$$

Similarly, (b) leads to $\sin(x - \frac{\pi}{4}) = -\dfrac{(4n + 1)\pi}{2\sqrt{2}}$, which is also impossible.

15. a) $y = \sin x$
[Also (h)]

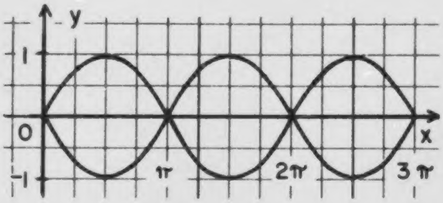


b) $y = \sqrt{1 - \cos^2 x} = \sqrt{\sin^2} x = |\sin x|$
[Also (c), (f), and (k)]



c) $y = |\sin x|$ [Same as (b)]

d) $y^2 = \sin^2 x$, $y = \pm \sin x$ [Also (e), (g), (ℓ)]



e) Same as (d)

f) $y = \sqrt{\dfrac{1 - \cos 2x}{2}} = \sqrt{\dfrac{1 - (1 - 2\sin^2 x)}{2}} = \sqrt{\sin^2 x}$

$= |\sin x|$

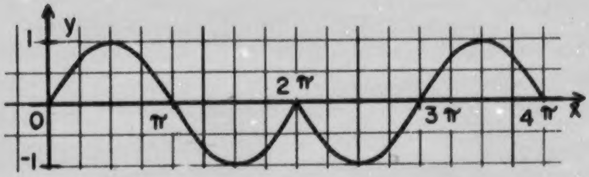
[Same as (b)]

g) $y^2 = \sin^2 x$ [See (d)]

h) $y = 2 \sin \frac{x}{2} \cos \frac{x}{2} = \sin x$ [See (a)]

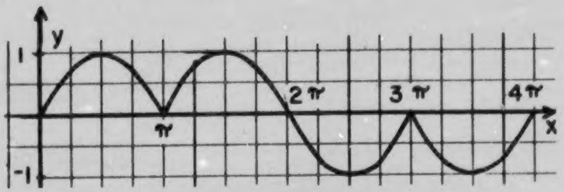
i) $y = 2\,|\sin \frac{x}{2}| \cos \frac{x}{2} = \quad = \sin x \dfrac{|\sin \frac{x}{2}|}{\sin \frac{x}{2}}$

$$= \begin{cases} \sin x & 0 \le x < 2\pi \\ -\sin x & 2\pi \le x < 4\pi \end{cases}$$



j) $y = 2 \sin \frac{x}{2}\, |\cos \frac{x}{2}| = \sin x \dfrac{|\cos x/2|}{\cos x/2}$

$$= \begin{cases} \sin x & 0 \le x < \pi \\ & 3\pi \le x < 4\pi \\ -\sin x, & \pi \le x < 3\pi \end{cases}$$



k) $y = 2\,|\sin \frac{x}{2} \cos \frac{x}{2}| = |\sin x|$ [See (b)]

ℓ) $y^2 = 4 \sin^2 \frac{x}{2} \cos^2 \frac{x}{2} = \sin^2 x$ [See (d)]

Illustrative Test Questions

1. Determine whether each of the following functions is periodic and, if so, find the fundamental period:
   (a) $y = |\cos 2x|$
   (b) $y = \sin 3x \cos 3x$
2. Given that $f: x \to f(x)$ is periodic with fundamental period $\frac{1}{2}$ and given that $f(\frac{1}{4}) = 2$, $f(2) = 5$, and $f(\frac{11}{8}) = 3$, find
   (a) $f(0)$
   (b) $f(-\frac{5}{8})$
   (c) $f(\frac{3}{4})$
3. Sketch two complete cycles of the graph of $y = 2 \sin 3x$.
4. Change from radians to degrees:
   (a) $\frac{7\pi}{12}$
   (b) $\frac{2}{15}$
5. Change from degrees to radians:
   (a) 165°
   (b) 2°
6. What is the radius of a circle in which a sector of area 6 has a perimeter 10?
   (two solutions)
7. Sketch the graph of $y = \sin x - \sqrt{3} \cos x$ over a complete cycle, indicating both the fundamental period and the amplitude.
8. Express $\sin(x + 2y)$ in terms of $\sin x$, $\sin y$, $\cos x$, $\cos y$.
9. Express the following in the form $\pm \sin x$ or $\pm \cos x$:
   (a) $\sin(x + \frac{3\pi}{2})$
   (b) $\cos(\frac{5\pi}{2} - x)$
   (c) $\sin(-3\pi - x)$
   (d) $\cos(x + 5\pi)$

10. Show that

$$(\sin x + \sin 2x)(\sin x)(1 - 2\cos x) = (\cos x + \cos 2x)(\cos 2x - \cos x)$$

holds for all real values of x.

11. Given $\sin 27^\circ = 0.4540$ and $\sin 28^\circ = 0.4695$, interpolate to find
    (a) $\sin 27.4^\circ$
    (b) the angle between 27° and 28° whose sine is 0.4664.

12. Given the function $x \to -3\sin\left(2x + \frac{\pi}{3}\right)$, find the points on the graph with smallest positive x for which
    (a) the function has the value zero
    (b) the function has a maximum value
    (c) the function has a minimum value

*13. If a, b, c are constants, find A and B such that $\sin(x + c) = A\sin(x + a) + B\sin(x + b)$ holds for all values of x. (You may assume that $\sin(a - b) \neq 0$.)

14. Evaluate (a) $\cos\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$
    (b) $\sin\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)$

*15. Consider the function $f : x \to \cos(\sin^{-1} x)$, $-1 \leq x \leq 1$.
    (a) Find an algebraic expression for $f(x)$.
    (b) What is the range of f?
    (c) Does f have an inverse?

16. Find the values of x in the interval $0 \leq x < 2\pi$ which satisfy
    (a) $\sin\left(3x + \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{3} - 2x\right)$
    (b) $\sin 2x - \cos 2x = \frac{\sqrt{6}}{2}$

Answers to Illustrative Test Questions

1. (a) $\cos 2\,(x + \frac{\pi}{2}) = \cos\,(2x + \pi) = -\cos 2x$

Hence, $|\cos 2\,(x + \frac{\pi}{2})| = |\cos 2x|$ and the period is $\frac{\pi}{2}$

(b) $\sin 3x \cos 3x = \frac{1}{2} \sin 6x = \frac{1}{2} \sin\,(6x + 2\pi)$

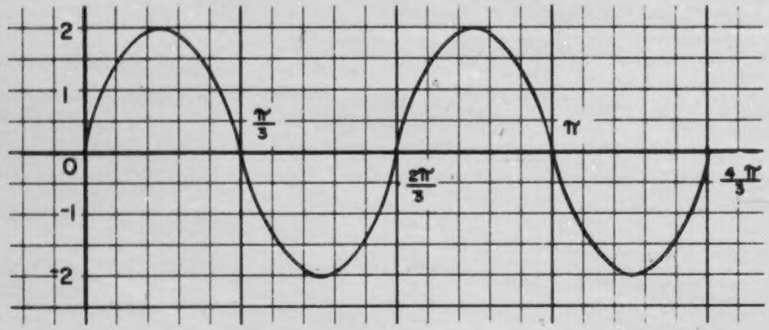
$= \frac{1}{2} \sin 6\,(x + \frac{\pi}{3})$ and the period is $\frac{\pi}{3}$.

2. (a) $f(0) = f(2 - 4\cdot\frac{1}{2}) = f(2) = 5$

(b) $f(-\frac{5}{8}) = f(\frac{11}{8} - 4\cdot\frac{1}{2}) = f(\frac{11}{8}) = 3$

(c) $f(\frac{3}{4}) = f(\frac{1}{4} + \frac{1}{2}) = f(\frac{1}{4}) = 2$.

3.



4. (a) $\frac{7\pi}{12}\cdot\frac{180^\circ}{\pi} = 105^\circ$ (b) $\frac{2}{15}\cdot\frac{180^\circ}{\pi} = \frac{24^\circ}{\pi}$

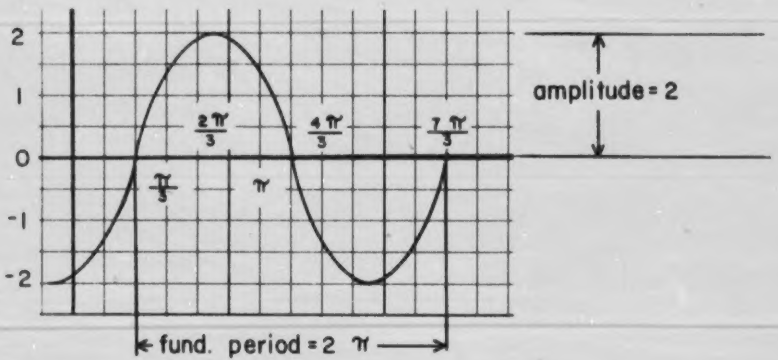
5. (a) $165^\circ\cdot\frac{\pi}{180^\circ} = \frac{11\pi}{12}$ (b) $2^\circ\cdot\frac{\pi}{180^\circ} = \frac{\pi}{90}$

6. If the radius is r and the arc s, then

$\frac{1}{2}r\,s = 6;$ $2r + s = 10;$ $\frac{1}{2}r\,(10 - 2r) = 6;$

$5r - r^2 = 6;$ $r^2 - 5r + 6 = 0;$ $(r - 3)(r - 2) = 0,\ r = 3$ or 2.

7. Note that $y = \sin x - \sqrt{3} \cos x = 2 \sin (x - \frac{\pi}{3})$ for all x.



8. $\sin (x + 2y) = \sin x \cos 2y + \cos x \sin 2y$

$$= \sin x (\cos^2 y - \sin^2 y) + \cos x \cdot 2 \sin y \cos y$$

$$= \sin x \cos^2 y - \sin x \sin^2 y + 2 \sin y \cos x \cos y$$

or $\sin x - 2 \sin x \sin^2 y + 2 \sin y \cos x \cos y$

9. (a) $-\cos x$
   (b) $\sin x$
   (c) $\sin x$
   (d) $- \cos x$

10. $(\sin x + \sin 2x)(\sin x)(1 - 2 \cos x)$

$= (\cos x + \cos 2x)(\cos 2x - \cos x)$

$(\sin x + \sin 2x)(\sin x - 2 \sin x \cos x)$

$(\sin x + \sin 2x)(\sin x - \sin 2x)$

$\sin^2 x - \sin^2 2x$

$1 - \cos^2 x - (1 - \cos^2 2x)$

$\cos^2 2x - \cos^2 x$

$(\cos 2x + \cos x)(\cos 2x - \cos x)$

11. (a)

$$\begin{array}{rr} \sin 28^\circ = & 0.4695 \\ \sin 27^\circ = & \underline{0.4540} \\ & 0.0155 \\ & \underline{.4} \\ & .00620 \\ & \underline{.4540} \\ \sin 27.4^\circ = & .4602 \end{array}$$

(b)

$$\begin{array}{r} .4664 \\ \underline{.4540} \\ .0124 \end{array} \qquad \frac{.0124}{.0155} = 0.8$$

$$\sin\ 27.8^\circ = 0.4664$$

12. (a) $- 3 \sin (2x + \frac{\pi}{3}) = 0$

$$2x + \frac{\pi}{3} = 0, \quad \pi, \quad - \pi, \ \ldots$$

$$2x = - \frac{\pi}{3}, \quad \frac{2\pi}{3}, \quad - \frac{4\pi}{3}, \ \ldots$$

$$x = - \frac{\pi}{6}, \quad \frac{\pi}{3}, \quad - \frac{2\pi}{3}, \ \ldots$$

Answer: $\frac{\pi}{3}$

(b) $- 3 \sin (2x + \frac{\pi}{3}) = + 3$

$$\sin (2x + \frac{\pi}{3}) = - 1$$

$$2x + \frac{\pi}{3} = \frac{3\pi}{2}, \ - \frac{\pi}{2}, \ \ldots$$

$$2x = \frac{7\pi}{6}, \ - \frac{5\pi}{6}, \ \ldots$$

$$x = \frac{7\pi}{12}, \ - \frac{5\pi}{12}, \ \ldots$$

Answer: $\frac{7\pi}{12}$

(c) $- 3 \sin (2x + \frac{\pi}{3}) = - 3$

$\sin (2x + \frac{\pi}{3}) = 1$

$2x + \frac{\pi}{3} = \frac{\pi}{2} , - \frac{3\pi}{2} , \ldots$

$2x = \frac{\pi}{6}, -\frac{11\pi}{6}, \ldots$

$x = \frac{\pi}{12} , - \frac{11\pi}{12} , \ldots$

Answer: $\frac{\pi}{12}$

13. If $x = - a$, we have

$$\sin (c - a) = A \sin 0 + B \sin (b - a)$$

$$B = \frac{\sin (c - a)}{\sin (b - a)} = \frac{\sin (a - c)}{\sin (a - b)} .$$

If $x = - b$, we have

$$\sin (c - b) = A \sin (a - b) + B \sin 0$$

$$A = \frac{\sin (c - b)}{\sin (a - b)} = \frac{\sin (b - c)}{\sin (b - a)} = - \frac{\sin (b - c)}{\sin (a - b)}$$

14. (a) $\cos (\sin^{-1}(- \frac{1}{2})) = \cos (- \frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

(b) $\sin (\cos^{-1}(- \frac{1}{2})) = \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

15. (a) $f(x) = \cos (\sin^{-1} x)$.

If we set $y = \sin^{-1} x$, then $x = \sin y$ and $f(x) = \cos y$. Hence $[f(x)]^2 + x^2 = \cos^2 y + \sin^2 y = 1$ and $[f(x)]^2 = 1 - x^2$. For $-1 \leq x \leq 1$, we have $- \frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $f(x) = \cos y \geq 0$. Hence

$f(x) = \sqrt{1 - x^2}$

(b) $0 \leq f(x) = \sqrt{1 - x^2} \leq 1$

(c) No. $f(- x) = f(x)$ for all x in the domain of f, hence f is not one-to-one and therefore does not have an inverse.

16. (a) Since $\sin(y + \frac{\pi}{2}) = \cos y$, we can transform the equation

to $\cos 3x = \cos(\frac{\pi}{3} - 2x)$

$\cos 3x - \cos(\frac{\pi}{3} - 2x) = 0$

Using (5) on Page 370, we obtain

$-2\sin(\frac{x}{2} + \frac{\pi}{6})\sin(\frac{5x}{2} - \frac{\pi}{6}) = 0$, hence

$\sin(\frac{x}{2} + \frac{\pi}{6}) = 0$ or $\sin(\frac{5x}{2} - \frac{\pi}{6}) = 0$

$\frac{x}{2} + \frac{\pi}{6} = 0, \ \pi, 2\pi, \ldots$ $\quad$ $\frac{5x}{2} - \frac{\pi}{6} = 0, \pi, 2\pi, 3\pi, \ldots$

$\frac{x}{2} = -\frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}\ldots,$ $\quad$ $\frac{5x}{2} = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}, \ldots$

$x = -\frac{\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{3}\ldots$ $\quad$ $x = \frac{\pi}{15}, \frac{7\pi}{15}, \frac{13\pi}{15}, \frac{19\pi}{15},$

$\frac{25\pi}{15}, \ldots$

and the admissable roots are

$\frac{5\pi}{3}$; $\frac{\pi}{15}$; $\frac{7\pi}{15}$; $\frac{13\pi}{15}$; $\frac{19\pi}{15}$

(b) Using the same transformations as in (a), we find

$\cos(\frac{\pi}{2} - 2x) - \cos 2x = \frac{\sqrt{6}}{2}$

$-2\sin\frac{\pi}{4}\ \sin(\frac{\pi}{4} - 2x) = \frac{\sqrt{6}}{2}$

$-\sqrt{2}\ \sin(\frac{\pi}{4} - 2x) = \frac{\sqrt{6}}{2}$

$\sin(\frac{\pi}{4} - 2x) = -\frac{\sqrt{3}}{2}$

$\sin(2x - \frac{\pi}{4}) = \frac{\sqrt{3}}{2}$

$2x - \frac{\pi}{4} = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \ldots$

$2x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{31\pi}{12}, \frac{35\pi}{12}, \ldots$

$x = \frac{7\pi}{24}, \frac{11\pi}{24}, \frac{31\pi}{24}, \frac{35\pi}{24}.$

2-11. Mathematical Induction

Both the teacher and the students can have a great deal of fun with this topic. The section should not be attempted with a below-average class. For an average class it is probably wise to eliminate the second principle of mathematical induction (and, of course, all exercises which depend upon it) as well as the Exercises 10 and 11 of false proofs by induction. The false propositions "proved" in these examples are deliberately outrageous on first sight so that even the poorest student will be aware that there is a flaw in the logic, whether or not he can find it. The flaw in Example 10 is, of course, that the initial step fails: 1 is not an even number. In Example 11, it is the sequential step that fails; in particular, A_1 does not imply A_2.

The exercises are graded. Exercises 1 - 12 may be attempted in any class (of course, omit Exercise 7 if you do not cover the second principle.) The starred exercises should be reserved for the best students. The double-starred exercises can be used to keep even the very brightest students busy. Be sure not to give away the show on the starred exercises. Their purpose is to develop originality. Let the student make a serious attack on the problem before giving any assistance. These are very difficult problems, and the student who makes some progress should be given a good deal of credit, even though he does not produce a complete solution.

Solutions to Exercises 2-11.

1. (First principle).

Initial Step: For $n = 1$, $\frac{1}{2} n (n + 1) = 1$.

Sequential Step: If the result is true for k, that is, if

$$1 + 2 + 3 + \cdots + k = \frac{1}{2} k (k + 1),$$

then

$$(1 + 2 + 3 + \cdots + k) + (k + 1) = \frac{1}{2} k (k + 1) + k + 1$$
$$= (k + 1) (\tfrac{1}{2}k + 1)$$
$$= \frac{1}{2} (k + 1) (k + 2)$$

q.e.d.

2a. (First principle)

Initial Step: For $n = 1$,

$$\frac{n}{2} [2a + (n - 1) d] = a.$$

Sequential Step: Denote the sum of the series to the first n terms by S_n. If the result is true for k then

$$S_{k+1} = S_k + (a + kd)$$
$$= \frac{k}{2} [2a + (k - 1)d] + a + kd$$
$$= \frac{1}{2} \{ [2ak + k(k - 1)d] + [2a + 2kd]\}$$
$$= \frac{1}{2} \{2a (k + 1) + kd [(k - 1) + 2]\}$$
$$= \frac{1}{2} \{2a (k + 1) + kd (k +1) \}$$
$$= \frac{1}{2} (k + 1) [2a + kd].$$

q.e.d.

2b. (First Principle)

Initial Step: For $n = 1$,

$$\frac{a (r^n - 1)}{r - 1} = a$$

provided $r \neq 1$. (Point out to the class that the sum formula is not valid when $r = 1$.)

Sequential Step: Denote the sum of the series to the first n terms by S_n. If the result is true for $n = k$, then

$$S_{k+1} = S_k + ar^k = \frac{a(r^k - 1)}{r - 1} + ar^k$$

$$= \frac{a(r^k - 1) + ar^k(r - 1)}{r - 1}$$

$$= \frac{a[\ (r^k - 1) + (r^{k+1} - r^k)]}{r - 1}$$

$$= \frac{a(r^{k+1} - 1)}{r - 1}.$$

q.e.d.

3. (First Principle)

Initial Step: For $n = 1$ we have

$$\frac{1}{3}(4n^3 - n) = \frac{1}{3}(4 - 1) = 1$$

Sequential Step: Denote the sum of the series to the first n terms by S_n. If the result is true for $n = k$, then

$$\begin{aligned} S_{k+1} &= S_k + [2(k + 1) - 1]^2 \\ &= \frac{1}{3}(4k^3 - k) + (2k + 1)^2 \\ &= \frac{1}{3}[\ (4k^3 - k) + 3(2k + 1)^2] \\ &= \frac{1}{3}[\ (4k^3 - k) + 3(4k^2 + 4k + 1)] \\ &= \frac{1}{3}[\ 4(k^3 + 3k^2 + 3k + 1) - k - 1] \\ &= \frac{1}{3}[\ 4(k+1)^3 - (k + 1)]. \end{aligned}$$

4. (First Principle)

Initial Step: $2 \cdot 1 = 2 = 2^1$.

Sequential Step: Let us assume the truth of the assertion for $n = k$, that is

$$2^k \geq 2k.$$

On multiplying by 2 we have

$$2 \cdot 2^k = 2^{k+1} \geq 2(2k) = 2k + 2k.$$

On the other hand k is a natural number and therefore $k \geq 1$. It follows that $2k \geq 2$ and $2k + 2k \geq 2k + 2$. Consequently,

$$2^{k+1} \geq 2k + 2k \geq 2k + 2 = 2(k+1).$$

q.e.d.

5. (First Principle)

Initial Step: If $n = 1$ we have

$$(1 + p)^n = 1 + p = 1 + np.$$

Sequential Step: Let us suppose the truth of the assertion for $n = k$, that is

$$(1 + p)^k \geq 1 + kp.$$

Since $p > -1$, $(1 + p)$ is positive, and we may multiply by $(1+p)$ without changing the sense of the inequality. It follows that

$$(1 + p)^{k+1} \geq (1 + p)(1 + kp),$$

but,

$$\begin{aligned}(1 + p)(1 + kp) &= 1 + p + kp + kp^2 \\ &= 1 + (k + 1)p + kp^2 \\ &\geq 1 + (k + 1)p,\end{aligned}$$

since kp^2 is non-negative. It follows that

$$(1 + p)^{k+1} \geq 1 + (k + 1)p.$$

q.e.d.

6. (First Principle)

Initial Step: For $n = 1$ the relation

$$1 = 1 + (n - 1)2^n.$$

is plainly satisfied.

Sequential Step:

Let S_n denote the sum of the series to n terms. If the theorem is true for $n = k$ then

$$\begin{aligned} S_{k+1} &= S_k + (k+1)\,2^k \\ &= 1 + (k-1)2^k + (k+1)2^k \\ &= 1 + [(k-1) + (k+1)]\,2^k \\ &= 1 + [2k]2^k \\ &= 1 + k2^{k+1}. \end{aligned}$$

q.e.d.

7a. (Second Principle)

The student may wonder that there is anything to prove here. If the question arises you might point out that a natural number may have composite factors and that these, in turn, may have composite factors, and so on. We could not be sure that we would ever get down to a factorization into primes.

Initial Step:

For $n = 1$ the number $n + 1 = 2$ is a prime.

Sequential Step:

Suppose the assertion to be true for all natural numbers less than or equal to k. For the number $k + 1$ there are two possibilities:

a) $k + 1$ is prime and the result is true.

b) $k + 1$ has a factor a which is neither 1 nor $k + 1$. In other words

$$k + 1 = ab$$

where a and b are natural numbers. From $a \neq 1$ we have $b \neq k + 1$, and from $a \neq k + 1$ we have $b \neq 1$. It follows for both that $1 < a, b \leq k$. Clearly, then a, b are either prime or factorable into primes and the desired factorization of $k + 1$ is obtained by forming their product.

7b. (Second Principle)

Initial Step: For $n = 1$, the number U_1 is defined to be $a = 1 \cdot a$.

Sequential Step: Suppose that the assertion is true for all natural numbers less than or equal to k. We know for U_{k+1} that there exists natural numbers p, q such that

$$p + q = k + 1 \text{ and } \quad U_{k+1} = U_p + U_q.$$

Since p is a natural number $p \geq 1$ and it follows that $q \leq k$. Similarly, from $q \geq 1$ it follows that $p \leq k$. But if p and q are less than or equal to k we have $U_p = pa$, $U_q = qa$ and

$$U_{k+1} = U_p + U_q = pa + qa = (p + q)a = (k + 1)a.$$

q.e.d.

N. B. The rare philosophically minded student may wonder how we could be sure beforehand that such numbers U_n exist. You might point out that what really has been proved is that if there exist numbers U_n with the properties described above then they must be the numbers we have obtained. It is an easy matter to verify that the properties are actually satisfied.

8. (First Principle)

The student should be expected to compile a table for a few values of the sum S_n to n terms:

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{2\cdot 3} = \frac{4}{2\cdot 3} = \frac{2}{3}$$

$$S_3 = \frac{2}{3} + \frac{1}{3\cdot 4} = \frac{9}{3\cdot 4} = \frac{3}{4}$$

$$S_4 = \frac{3}{4} + \frac{1}{4\cdot 5} = \frac{16}{4\cdot 5} = \frac{4}{5}$$

$$\cdots$$

At this point he will be well on his way to drawing the correct hypothesis and to establishing his proof by induction.

Hypothesis: For all natural numbers n the sum S_n to n terms of the given series satisfies

$$S_n = \frac{n}{n + 1}$$

[sec. 2-11]

Proof: The initial step is already verified. For the sequential step we assume that the theorem is true for $n = k$. We then have

$$\begin{aligned}
S_{k+1} &= S_k + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{(k+2)k+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2} \quad . \qquad \text{q.e.d.}
\end{aligned}$$

After the student has worked through this problem systematically, you may wish to point out a quick proof using the fact that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}, \text{ so that}$$

$$S_n = (1 - \tfrac{1}{2}) + (\tfrac{1}{2} - \tfrac{1}{3}) + \dots + (\tfrac{1}{n} - \tfrac{1}{n+1}) = 1 - \frac{1}{n+1} = \frac{n}{n+1} .$$

The only trouble with this is that the student may fail to realize that a proof by mathematical induction is still necessary.

9. (First Principle)

There are many possible attacks on this problem. Here we observe on tabulating a few values of the sum S_n to n terms:

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + 8 = 9 = 3^2 \\
S_3 &= 9 + 27 = 36 = 6^2 \\
S_4 &= 36 + 64 = 100 = 10^2 \\
S_5 &= 100 + 125 = 225 = 15^2 \\
&\dots
\end{aligned}$$

We conjecture that S_n is a perfect square, in general, and seek a pattern to describe the numbers

$$1, 3, 6, 10, 15, \ldots$$

We may observe that

$$\begin{aligned} 1 &= 1 \\ 3 &= 1 + 2 \\ 6 &= 1 + 2 + 3 \\ 10 &= 1 + 2 + 3 + 4 \\ 15 &= 1 + 2 + 3 + 4 + 5. \end{aligned}$$

From the result of Exercise 1 we therefore conjecture that

$$S_n = \frac{1}{4} n^2(n + 1)^2.$$

Proof:

Initial Step:

$S_1 = 1$ is satisfied.

Sequential Step: If the hypothesis is true for $n = k$ then

$$\begin{aligned} S_{k+1} &= S_k + (k + 1)^3 \\ &= \frac{1}{4} k^2(k + 1)^2 + (k + 1)^3 \\ &= \frac{1}{4} [k^2 (k + 1)^2 + 4(k + 1)^3] \\ &= \frac{1}{4} (k + 1)^2[k^2 + 4(k + 1)] \\ &= \frac{1}{4} (k + 1)^2(k + 2)^2. \end{aligned}$$

q.e.d.

10. (First Principle)

There are many ways of doing this problem. Perhaps the simplest is to observe that the n-th term is simply $n^2 + n$ so that the sum S_n of this series is obtained simply by adding the result of Exercise 1 to the sum of the sequence of squares obtained in Example 7. We have, then

$$S_n = \frac{1}{2} n(n + 1) + \frac{1}{6} n (n + 1) (2n + 1)$$
$$= \frac{1}{6} n(n + 1) (2n + 4)$$
$$= \frac{1}{3} n(n + 1) (n + 2).$$

To prove this result by induction we observe first that the initial step is correct. Next, if the result is correct for $n = k$, then

$$S_{k + 1} = S_k + (k + 1) (k + 2)$$
$$= \frac{1}{3} k (k + 1) (k + 2) + (k + 1) (k + 2)$$
$$= \frac{1}{3} (k + 1) (k + 2) (k + 3).$$

q.e.d.

You might remark as an interesting sidelight that this result also demonstrates that the product of three consecutive natural numbers is divisible by 3.

11. (First Principle)

Let A_k be the assertion that for any $k + 2$ points

$P_1, P_2, \ldots, P_{k + 2}$,

$$S_k = m(P_1P_2) + m(P_2P_3) + \ldots + m(P_{k+1}P_{k+2}) \geq m(P_1P_{k+2}).$$

Initial Step:

A_1 is certainly true since it merely states the triangle inequality.

Sequential Step:

If A_k is true then

$$S_{k+1} = S_k + m(P_{k+2}P_{k+3})$$
$$\geq m(P_1P_{k+2}) + m(P_{k+2}P_{k+3}) \geq m(P_1P_{k+3}),$$

where the last step follows from the triangle inequality.

q.e.d.

12. (First Principle)

Initial Step:

For $n = 1$ the product contains just one term

$$(1 + \frac{3}{1}) = 4 = (1 + 1)^2.$$

Sequential Step:

If the assertion is true for $n = k$ then, denoting the product of the k factors by P_k we have

$$\begin{aligned} P_{k+1} &= P_k\left(1 + \frac{2k + 3}{(k + 1)^2}\right) \\ &= (k + 1)^2\left(1 + \frac{2k + 3}{(k + 1)^2}\right) \\ &= (k + 1)^2 + (2k + 3) \\ &= (k^2 + 2k + 1) + (2k + 3) \\ &= k^2 + 4k + 4 \\ &= (k + 2)^2. \end{aligned}$$

q.e.d.

13. (Second Principle)

Let $U_n = n(n^2 + 5)$ for all natural numbers n. We note that U_n is a cubic polynomial in n. It is easy enough to see that the difference

$$V_n = U_n - U_{n-1}$$

is a quadratic polynomial in n, and similarly that

$$W_n = V_n - V_{n-1} = U_n - 2U_{n-1} + U_{n-2}$$

is a linear polynomial in u. This suggests that we form successive differences in this way to obtain something simpler to work with. If it turns out that V_n is divisible by 6 then we may use the first principle to show that

$$U_n = V_n + U_{n-1}$$

is divisible by 6. If not, we may go on to the next difference. If W_n is divisible by 6 then to obtain the desired result from

$$U_n = W_n + 2U_{n-1} - U_{n-2}$$

we must use the second principle. If W_n is not divisible by 6 we continue in the same way. This technique works for the most complicated of this kind of divisibility problem and you may wish to suggest to the student that he invent his own problems.

First we observe that $U_1 = 6$ so that the result is certainly correct in that case. Next we assume the result is true for all natural numbers less than or equal to k. We have

$$\begin{aligned} V_{k+1} &= U_{k+1} - U_k = (k+1)\ [(k+1)^2 + 5] - k[k^2 + 5] \\ &= [(k+1)^3 - k^3] + 5(k+1) - 5k \\ &= 3k^2 + 3k + 6 \\ &= 3(k^2 + k + 2). \end{aligned}$$

(It follows from the first principle that U_{k+1} is divisible by 3.)

Proceeding one more step we have

$$\begin{aligned} W_{k+1} &= V_{k+1} - V_k = U_{k+1} - 2U_k + U_{k-1} \\ &= 3\ \{[k^2 + k + 2] - [(k-1)^2 + (k-1) + 2]\} \\ &= 6k. \end{aligned}$$

q.e.d.

It follows that

$$U_{k+1} = 2U_k - U_{k-1} + 6k. \qquad (1)$$

Hence, since U_k and U_{k-1} are divisible by 6 we see that U_{k+1} is divisible by 6.

It is good to point out to the student that the proof is not complete at this point since the argument going from k to k + 1 must be valid for all k. In this case equation (1) is meaningless for k = 1, since U_0 is undefined (U_n is defined only for natural numbers n). There are two ways to surmount this difficulty.

a. Simply extend the interpretation of the formula for U_n so that $U_0 = 0$. The method of backward extension is often quite useful.

b. Let the k-th assertion A_k be that both U_k and U_{k+1} are divisible by 6 (rather than just U_k.) To prove the general result it then becomes necessary to establish both $U_1 = 6$ and $U_2 = 6\cdot 3$ as special cases in the initial step. (This method is quite general. Note that it is similar to a proof of the second principle by the first.)

*14. (Second Principle)

This is a problem for which the methods of proof are diverse. Most usually, the student will probably discover that the payments are in arithmetic progression beginning at the leader, going around the circle and returning to the leader again. He will then realize that all payments must be equal. Here is another approach.

Let us suppose there are n pirates in addition to the leader. We assume $n > 1$; otherwise the result is obvious. Let P_0 be the amount of payment to the leader and let P_1, P_2, ... , P_n be the payments to the other pirates going to the right from the leader around the circle. Except for the leader, we know that each pirate receives a payment equal to the average of the two men on his right and left. It follows that for $1 \leq k \leq n - 1$ we have

(1)
$$P_k = \frac{P_{k-1} + P_{k+1}}{2} \quad (k = 1, \dots, n - 1),$$

and for $k = n$

(2)
$$P_n = \frac{P_{n-1} + P_0}{2}.$$

We consider three cases: P_1 is equal to, is greater than or is less than P_0.

a. Suppose $P_1 = P_0$. Then from

$$P_1 = \frac{P_0 + P_2}{2} = P_0$$

we have

$$P_2 = P_0.$$

Now if it is true that $P_k = P_{k-1}$ we have, following the same line of argument, that

$$P_k = \frac{P_{k+1} + P_{k-1}}{2} = P_{k-1}$$

and, therefore,

$$P_{k+1} = P_k.$$

It follows by the second principle (if for all natural numbers j less than or equal to k the values of P_k are all equal to P_0 then $P_{k+1} = P_0$) that, in so far as formula (1) holds, all values of $P_k = P_0$. In other words

$$P_k = P_0, \text{ for } k = 1, \ldots, n - 1.$$

For $k = n$ it follows from (2) that

$$P_n = \frac{P_0 + P_0}{2} = P_0.$$

We see then that if the man on the leader's right gets the same amount as the leader, so does everyone else.

b. Suppose $P_1 < P_0$. Then from

$$P_1 = \frac{P_0 + P_2}{2}$$

we have

$$P_2 = 2P_1 - P_0 = P_1 + P_1 - P_0 < P_1 + P_0 - P_0 = P_1 < P_0$$

Assuming that $P_j < P_{j-1} < P_0$ for all natural numbers j such that $j \leq k$ we see by applying the same argument to

$$P_{k+1} = 2P_k - P_{k-1}$$

that $P_{k+1} < (P_k + P_{k-1}) - P_{k-1} < P_k < P_0$ insofar as formula (1) holds. For $k = n$ we have, in particular

$$P_n < P_{n-1} < P_0.$$

But, from (2)

$$P_0 = 2P_n - P_{n-1} < (P_n + P_{n-1}) - P_{n-1} < P_n < P_0,$$

a contradiction. We conclude that the man on the right cannot receive less than the leader.

c. $P_1 > P_0$. By an argument exactly parallel to that of (b) it can be shown that this is not possible.

Conclusion: Combining the results a, b, c, we see that the loot may only be divided into equal parts.

*15. (First Principle)

First we begin with the proof that p_n and q_n are relatively prime, that is p_n and q_n have no common factor greater than one.

Initial Step: The assertion is true for $n = 1$.

Sequential Step: Suppose the assertion is true for $\frac{p_k}{q_k}$. We prove it for $\frac{p_{k+1}}{q_{k+1}}$. From

$$p_{k+1} = p_k + 2q_k; \qquad q_{k+1} = p_k + q_k$$

we obtain

$$p_k = 2q_{k+1} - p_{k+1},$$

$$q_k = p_{k+1} - q_{k+1}$$

It follows at once that any common factor of p_{k+1} and q_{k+1} is a common factor of p_k and q_k. Since 1 is the greatest common divisor or p_k and q_k, it must also be the greatest common divisor of p_{k+1} and q_{k+1}.

q.e.d.

For the purpose of answering the rest of the question we define the error at the n-th stage of approximation as

$$e_n = \frac{p_n}{q_n} - \sqrt{2}.$$

Now, let us attempt to represent e_{k+1} in terms of e_k. We have

$$\begin{aligned} e_{k+1} &= \frac{p_{k+1}}{q_{k+1}} - \sqrt{2} = \frac{p_k + 2q_k}{p_k + q_k} - \sqrt{2} \\ &= \frac{(p_k/q_k) + 2}{(p_k/q_k) + 1} - \sqrt{2} \\ &= \frac{(e_k + \sqrt{2}) + 2}{(e_k + \sqrt{2}) + 1} - \sqrt{2} \\ &= \frac{(e_k + \sqrt{2} + 2) - \sqrt{2}(e_k + \sqrt{2} + 1)}{(e_k + \sqrt{2} + 1)} \\ &= \frac{(1 - \sqrt{2})\, e_k + (\sqrt{2} + 2) - (2 + \sqrt{2})}{e_k + \sqrt{2} + 1} \\ &= \frac{(1 - \sqrt{2})e_k}{e_k + \sqrt{2} + 1}. \end{aligned}$$

In order to simplify the work we multiply the numerator and denominator by $1 + \sqrt{2}$ to obtain

$$e_{k+1} = \frac{-e_k}{(\sqrt{2} + 1)^2 + e_k(\sqrt{2} + 1)} \qquad (1)$$

From this result we shall obtain all we need. Two things are clear from (1). If e_k is small enough, it will have little effect in the denominator; the denominator will be positive, even greater than one, and e_{k+1} will have opposite sign from e_k.

First we observe that $|e_1| = |1 - \sqrt{2}| < 1$. If $|e_k| < 1$ we show that $|e_{k+1}| < 1$. In order to prove this we observe for the denominator D_{k+1} in (1) that

$$\begin{aligned} D_{k+1} &= (\sqrt{2} + 1)^2 + e_k(\sqrt{2} + 1) = (\sqrt{2} + 1)\,[\sqrt{2} + 1 + e_k] \\ &= (\sqrt{2} + 1)\,[\sqrt{2} + (1 + e_k)\,]. \end{aligned}$$

Since $1 + e_k > 0$ it follows that

(2) $D_{k+1} > \sqrt{2}\,(\sqrt{2} + 1) = 2 + \sqrt{2} > 2 > 1.$

Entering this result in (1) we see that

$$(3) \qquad |e_{k+1}| = \frac{|e_k|}{D_{k+1}} < |e_k| < 1.$$

We have proved by mathematical induction that $|e_n| < 1$ for all natural numbers n but we have proved more. Since the denominator D_{n+1} is positive it follows immediately from (1) that the error alternates in sign, since $e_{n+1} = -e_n/D_{n+1}$. We have now only to prove that the error can be made as small as desired. In fact, we shall prove that

$$|e_{n+1}| < \frac{1}{2^n} \qquad (n = 1, 2, 3, \ldots).$$

Initial Step: For $n = 1$, we have

$$|e_2| = \frac{|e_1|}{D_2} = \frac{\sqrt{2} - 1}{(\sqrt{2} + 1)[(\sqrt{2} + 1) + (1 - \sqrt{2})]}$$

$$= \frac{\sqrt{2} - 1}{2(\sqrt{2} + 1)} = \frac{1}{2}\left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right) < \frac{1}{2} \cdot 1$$

$$< \frac{1}{2}.$$

Sequential Step:

From (2) above we have proved $D_n > 2 + \sqrt{2} > 2$ for all n. If

$$|e_{k+1}| < \frac{1}{2^k}$$

then

$$|e_{k+2}| = \frac{|e_{k+1}|}{D_{k+1}} < \frac{1}{2^k D_{k+2}} < \frac{1}{2^{k} \cdot 2}$$

$$< \frac{1}{2^{k+1}}.$$

With this result, the proof is complete.

*16. (First and Second Principles)

In solving this problem, as in many other mathematical problems, it pays to turn things around. Instead of thinking of $q(n)$ as the sum of $p(1) + p(2) + \ldots + p(n)$, we may think of $p(n)$ as the difference

$$p(n) = q(n + 1) - q(n). \qquad (1)$$

This relation suggests a converse to the theorem we wish to prove; namely, that if q is a polynomial function of degree $m + 1$ then there is a polynomial function of degree m for which (1) holds. We have already made use of this theorem in the solution to Exercise 13.

With this idea in mind, we are led to the following attack on the original problem: take a polynomial function of degree $m + 1$ (preferably the simplest one, $u: x \longrightarrow x^{m+1}$), form the difference

$$r(n) = u(n) - u(n - 1), \qquad (2)$$

and compare $r(n)$ with $p(n)$. We note that

$$\begin{aligned} r(1) &= u(1) - u(0) \\ r(2) &= u(2) - u(1) \\ &\ \ \vdots \\ r(n) &= u(n) - u(n - 1) \end{aligned}$$

and hence, adding

$$\begin{aligned} r(1) + r(2) + \ . \ . \ . + r(n) &= u(n) - u(0) \\ &= u(n) \qquad (3) \end{aligned}$$

since $u(0) = 0^{m+1} = 0$. (The result (3), though obvious, should itself be proved by mathematical induction; a proof by the first principle is easy.)

Our assertion A_k is: if p_k is a polynomial function of degree k, then there exists a polynomial function q of degree $k + 1$ such that

$$q(n) = p_k(1) + p_k(2) + \ . \ . \ . + p_k(n)$$

for each natural number n.

Initial Step:

In this case, $p_1(x) = A + Bx$, and the sum

$$p_1(1) + p_1(2) \ . \ . \ . + p_1(n)$$

is an arithmetic progression with first term $A + B$ and common difference B.

Hence, by Exercise 2,

$$\begin{aligned} p_1(1) + p_1(2) + \ . \ . \ . + p_1(n) &= \frac{n}{2}[2(A + B) + (n - 1)B] \\ &= \frac{B}{2}n^2 + (A + \frac{B}{2})n, \end{aligned}$$

and the polynomial function $q: x \to \frac{B}{2}x^2 + (A + \frac{B}{2})x$,

of degree 2, has the required property.

Sequential Step: We suppose that the assertions $A_1, A_2, \ldots, A_k$ are all true, that is, the result is proved for polynomials of degree at most k.

We may write any polynomial of degree $k + 1$ in the form

$$p_{k+1}(x) = ax^{k+1} + S_k(x) \qquad (a \neq 0) \qquad (4)$$

where $S_k(x)$ is of degree k at most. Since, by our induction hypothesis, the sum

$$S_k(1) + S_k(2) + \ldots + S_k(n)$$

is the value at $x = n$ of a polynomial of degree at most $k + 1$, we need not concern ourselves with the contribution of S_k and can devote our attention primarily to the term ax^{k+1}. We set

$$q(n) = p_{k+1}(1) + p_{k+1}(2) + \ldots + p_{k+1}(n),$$

and, using (4), we find

$$q(n) = a\left[1^{k+1} + 2^{k+1} + \ldots + n^{k+1}\right] + [S_k(1) + S_k(2) + \ldots + S_k(n)] \qquad (5)$$

We wish to compare this with the sum obtained in (3). We therefore define, by analogy with (2),

$$r_m(x) = x^{m+1} - (x-1)^{m+1}. \qquad (6)$$

If we expand $(x - 1)^{m+1}$ by the Binomial Theorem and combine terms, we obtain

$$\begin{aligned} r_m(x) &= x^{m+1} - \left[x^{m+1} - (m+1)x^m + \ldots + (-1)^{m+1}\right] \\ &= (m+1)x^m + t_{m-1}(x). \end{aligned} \qquad (7)$$

where t_{m-1} is a polynomial of degree $m - 1$.

Because the Binomial Theorem also demands proof by induction, we include a special proof of (7) at the end of this discussion of Exercise 16.

We have, from (3) and (6),

$$r_m(1) + r_m(2) + \ldots + r_m(n) = n^{m+1} \qquad (8)$$

From (7),

$$x^{m} = \frac{1}{m+1}[r_m(x) - t_{m-1}(x)]$$

and, setting $m = k + 1$, we get

$$x^{k+1} = \frac{1}{k+2}[r_{k+1}(x) - t_k(x)].$$

Substituting this result in (4), we have

$$p_{k+1}(x) = \frac{a}{k+2}[r_{k+1}(x) - t_k(x)] + S_k(x)$$

$$= br_{k+1}(x) + v_k(x) \qquad (9)$$

where $b = \frac{a}{k+2} \neq 0$ and $v_k(x) = S_k(x) - \frac{a}{k+2} t_k(x)$ is of degree at most k. We now substitute (9) in (5) getting

$$q(n) = b[r_{k+1}(1) + r_{k+1}(2) + \ldots + r_{k+1}(n)]$$

$$+[v_k(1) + v_k(2) + \ldots + v_k(n)]$$

$$= bn^{k+2} + [v_k(1) + v_k(2) + \ldots + v_k(n)].$$

by (8). Our induction hypothesis asserts that

$$v_k(1) + v_k(2) + \ldots + v_k(n) = w_{k+1}(n).$$

where w_{k+1} is a polynomial of degree $k + 1$ at most. Hence

$$q(x) = bx^{k+2} + w_{k+1}(x)$$

is a polynomial of degree $k + 2$, and the induction is complete.

We must now prove (7)

$$r_m(x) = x^{m+1} - (x-1)^{m+1} = (m+1)x^m + t_{m-1}(x).$$

Initial Step:

If $m = 1$, we have

$$x^2 - (x-1)^2 = 2x - 1.$$

Sequential Step:

Assume the result is true for $m = k$. At the $(k + 1)$th stage, we have

$$\begin{aligned} r_{k+1}(x) &= x^{k+2} - (x-1)^{k+2} \\ &= x^{k+2} + (x-1)[x^{k+1} - (x-1)^{k+1} - x^{k+1}] \\ &= x^{k+1}[x - (x-1)] + (x-1)r_k(x) \\ &= x^{k+1} + (x-1)\,[(k+1)x^k + t_{k-1}(x)\,] \\ &= x^{k+1} + (k+1)x^{k+1} - (k+1)x^k + (x-1)t_{k-1}(x) \\ &= (k+2)\,x^{k+1} + t_k(x) \end{aligned}$$

where $t_k(x) = -(k+1)x^k + (x-1)t_{k-1}(x)$ is a polynomial of degree at most k. This is the desired result.

**17.a (First Principle)

It is easy to see that we can never have $f(m) = g(n)$, since $9 = 3^2$ and consequently $g(n)$ is an even power of 3 for all natural numbers n, while $f(n)$ is always an odd power of 3. We must therefore show only that $m = n + 1$ is the least natural number m such that $f(m) > g(n)$. It is convenient to break this into two parts

a) $f(n+1) > g(n)$, and
b) $f(r) < g(n)$ for all $r \le n$.

We first examine (a): $f(n+1) > g(n)$. (1)

This means
$$3^{f(n)} > 9^{g(n-1)} = 3^{2g(n-1)} \quad (n>1)$$

or, since $x \longrightarrow 3^x$ is a strictly increasing function of x,
$$f(n) > 2g(n-1)$$
or, since $f(n)$ and $g(n-1)$ are natural numbers
$$f(n) \ge 2g(n-1) + 1. \qquad (2)$$
But (2) implies
$$3^{f(n)} \ge 3^{2g(n-1)+1} = 3 \cdot 9^{g(n-1)}$$
or
$$f(n+1) \ge 3\,g(n) = 2g(n) + g(n)$$
or, since $g(n)$ is a natural number and therefore $g(n) \ge 1$
$$f(n+1) \ge 2g(n) + 1. \qquad (3)$$
We have thus proved that (2) is equivalent to (1) and implies (3).

This is all we need to prove (a). Let our assertion A_n be $f(n + 1) > g(n)$.

Initial Step:

If $n = 1$, we have

$$f(2) = 27 \geq 2g(1) + 1 = 2 \cdot 9 + 1 = 19.$$

This implies, as we have shown, that

$$f(2) > g(1)$$

and A_1 is thus verified.

Sequential Step:

We assume the truth of A_k, which asserts that

$$f(k + 1) > g(k).$$

As we have seen, this is equivalent to

$$f(k) \geq 2g(k - 1) + 1$$

which implies

$$f(k + 1) \geq 2g(k) + 1$$

which is equivalent to

$$f(k + 2) > g(k + 1).$$

Thus A_k implies $A_{k + 1}$, and the inductive proof of part (a) is complete.

We must now prove part (b): $f(r) < g(n)$ for all $r \leq n$. Since, as has been observed, $x \longrightarrow 3^x$ is a strictly increasing function of x, it suffices to prove the case $r = n$.

Initial Step:

$f(1) = 3 < g(1) = 9.$

Sequential Step: Assume $f(k) < g(k)$. Then, since $g(k) \geq 1$, $f(k) < g(k) + g(k) = 2g(k)$.
But $f(k + 1) = 3^{f(k)}$, and $g(k + 1) = 9^{g(k)} = 3^{2g(k)}$, and therefore $f(k + 1) < g(k + 1)$, which completes the induction for part b) and thus the proof of the theorem.

**18. (First Principle)

Following the hint we obtain by experiment

$$x^n - y^n = (x + y)(x^{n-1} - y^{n-1}) - xy(x^{n-2} - y^{n-2}) \qquad (1)$$

Set

$$x = 1 + \sqrt{5}, \qquad y = 1 - \sqrt{5} \qquad (2)$$

and

$$I_n = (x^n - y^n) / 2^n \sqrt{5}. \qquad (3)$$

Using (1) and (2) in (3) we obtain

$$I_n = \frac{(x + y)\, 2^{n-1} \sqrt{5} \cdot I_{n-1} - xy \cdot 2^{n-2} \sqrt{5} \cdot I_{n-2}.}{2^n \sqrt{5}}$$

From $x + y = 2$, $xy = -4$ we then have

$$I_n = \frac{2^n \sqrt{5}\, I_{n-1} + 2^n \sqrt{5}\, I_{n-2}}{2^n \sqrt{5}}$$

$$= I_{n-1} + I_{n-2}.$$

Consequently, if I_{n-1} and I_{n-2} are integers so is I_n.

In order to frame a proof by mathematical induction we use the first principle and take for the assertion A_k that both I_k and I_{k+1} are integers.

Initial Step:

$$I_1 = \frac{(1 + \sqrt{5}) - (1 - \sqrt{5})}{2 \sqrt{5}} = 1,$$

$$I_2 = \frac{(6 + 2 \sqrt{5}) - (- 2 \sqrt{5})}{4 \sqrt{5}} = 1.$$

Sequential Step:

We assume that the theorem is true for $n = k$. We have by the argument above.

$$I_{k+2} = I_{k+1} + I_k.$$

The assertion A_k which we have assumed true is that I_k and I_{k+1} are integers. Consequently their sum I_{k+2} is an integer.

q.e.d.

(We could have used the second principle, pointing out that $I_{k+1} = I_k + I_{k-1}$ breaks down for $k = 1$, but that the result that I_2 is an integer holds anyway.

2-12. Solutions.

1. Eliminate a_0 by subtraction in pairs obtaining

$$+3a_1 - 3a_2 + 9a_3 = +1$$

$$+a_1 + 3a_2 + 7a_3 = -4$$

$$+2a_1 + 12a_2 + 56a_3 = +2$$

Eliminate a_1

$$12a_2 + 12a_3 = -13$$
$$6a_2 + 42a_3 = 10$$

Then

$$-72a_3 = -33$$
$$a_3 = +\frac{33}{72} = +\frac{11}{24}$$

Since $a_2 + a_3 = \frac{-13}{12}$

$$a_2 = -\frac{13}{12} - \frac{11}{24} = -\frac{37}{24}$$

$$a_1 = -3a_2 - 7a_3 - 4$$
$$= \frac{1}{24}[3.37 - 7.11 - 4.24]$$
$$= -\frac{62}{24} = \frac{-31}{12}$$

Finally, $a_0 = 3 - a_1 - a_2 - a_3 = \frac{160}{24} = \frac{20}{3}$

2. $\frac{-2}{15}x(x-2)(x-4) + \frac{1}{12}(x+1)x(x-4) + \frac{1}{20}(x+1)x(x-2)$

$$= \frac{x(x-4)}{3}\left[\frac{-2}{5}(x-2) + \frac{1}{4}(x+1)\right] + \frac{1}{20}(x+1)\,x\,(x-2)$$
$$= \frac{x(x-4)}{3}\left[\left(^{-}\frac{2}{5} + \frac{1}{4}\right)x + \left(\frac{4}{5} + \frac{1}{4}\right)\right] + \frac{1}{20}(x+1)x(x-2)$$
$$= \frac{x(x-4)}{60}(-3x + 21) + \frac{1}{20}(x+1)x(x-2)$$
$$= \frac{x(x-4)}{20}(-x + 7) + \frac{1}{20}(x+1)x(x-2)$$
$$= \frac{x}{20}[(x-4)(-x+7) + (x+1)(x-2)]$$
$$= \frac{x}{20}[10x - 30] = \frac{x}{2}(x-3)$$

Answer: $\frac{x(x-3)}{2}$

Check: $(0, 0)$: $\frac{0(-3)}{2} = 0$

$(-1, 2)$: $\frac{(-1)(-4)}{2} = 2$

$(2, -1)$: $\frac{2(-1)}{2} = -1$

$(4, 2)$: $\frac{4(1)}{2} = 2$

3. $(-1, 2), (0, -1), (2, 3)$

$$g_1(x) = 2\,\frac{(x - 0)\,(x - 2)}{(-1 - 0)\,(-1 - 2)} = \frac{2x(x - 2)}{3}$$

$$g_2(x) = -1\frac{(x + 1)\,(x - 2)}{(0 + 1)\,(0 - 2)} = \frac{1}{2}\,(x + 1)\,(x - 2)$$

$$g_3(x) = 3\,\frac{(x + 1)\,(x - 0)}{(2 + 1)\,(2 - 0)} = \frac{1}{2}\,x(x + 1)$$

$$g(x) = g_1(x) + g_2\,(x) + g_3\,(x)$$

$$= \frac{5x^2 - 4x - 3}{3}$$

Check: $g(-1) = 2$

$g(0) = -1$

$g(2) = 3$

4. A solution by Lagrange's Formula

$(-1, 6), (0, 1), (1, 0), (2, 9), (3, 34)$

$$f(x) = 6\;\frac{x(x-1)(x-2)(x-3)}{(-1)(-1-1)(-1-2)(-1-3)} + 1\;\frac{(x+1)(x-1)(x-2)(x-3)}{(1)\;(-1)\;(-2)\;(-3)} + 0$$

$$+9\,\frac{(x+1)(x)(x-1)(x-3)}{(2+1)(2)(2-1)(2-3)} + 34\frac{(x+1)(x)(x-1)(x-2)}{(3+1)(3)(3-1)(3-2)}$$

$$= \frac{1}{4}(x^4 - 6x^3 + 11x^2 - 6x) - \frac{1}{6}\,(x^4 - 5x^3 + 5x^2 - 6)$$

$$-\frac{3}{2}\,(x^4 - 3x^3 - x^2 + 3x) + \frac{17}{12}\,(x^4 - 2x^3 - x^2 + 2x)$$

$$= x^3 + 2x^2 - 4x + 1$$

Exercises 3-13. Solutions.

1. a) $x \rightarrow 2x - 1$
   b) $x \rightarrow 2x - 1$
   c) $x \rightarrow 4x^3 - 9x^2 + 1$
   d) $x \rightarrow 4x^3 - 9x^2 + 1$

2. a) $x \rightarrow x^2$ or $x \rightarrow x^2 + 1$
   b) $x \rightarrow x^3$ or $x \rightarrow x^3 + 1$
   c) $x \rightarrow x^6 - x^3 + 8x$ or $x \rightarrow x^6 - x^3 + 8x + 1$
   d) $x \rightarrow \frac{x^5}{5} + \frac{x^3}{3} + x$ or $x \rightarrow \frac{x^5}{5} + \frac{x^3}{3} + x + 1$

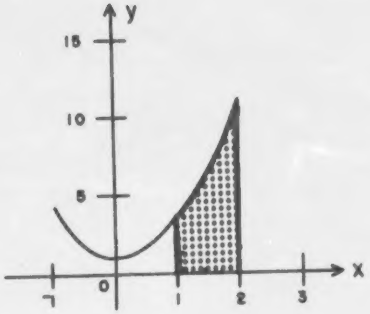
3. They differ by a constant.

4. a) $\frac{x^3}{3}$ at $x = 2$, or $8/3$.

   b) $x^2 + x$ at $x = 2$, or 6

   c) $x^4 + \frac{x^2}{2}$ at $x = 2$, or 18

5.



(Note different scales)

   b) $x^3 + x$ at $x = 1$ or 2
   c) $x^3 + x$ at $x = 2$ or 10
   d) $10 - 2 = 8$

6. $[(16x - \frac{x^3}{3})$ at $x = 3] - [(16x - \frac{x^3}{3})$ at $x = 2]$

   or $39 - (32 - 8/3) = 7 + 8/3 = 29/3$.

7. $[(x^4 - \frac{x^2}{2})$ at $x = 2] - [(x^4 - \frac{x^2}{2})$ at $x = 1]$

   $= 14 - 1/2 = 13\ 1/2$

8. a) $g(x) = 2x^3 + 2x$ or $g(x) = 2x^3 + 2x + c$
   $g(2) - g(0) = 20 - 0 = 20$ or $g(2) - g(0) = (20 + c) - c$
   $= 20$
   b) 20.

9. a) $g(x) = 2x^2 + 3x$ or $g(x) = 2x^2 + 3x + c$

   $g(2) - g(1) = 14 - 5 = 9$

   or $g(2) - g(1) = (14 + c) - (5 + c) = 9$.
   b) 9

10. Since $g(x) = h(x) + c$ where c is a constant.
    $g(5) - g(3) = [h(5) + c] - [h(3) + c]$

    $= h(5) - h(3)$

Answers to Exercises 4-15a. Page A-43.

1. $f(x + y) = 0$, $f(x) = 0$, $f(y) = 0$; hence (6) becomes $0 = 0 \cdot 0$. This function does not satisfy (3) because division by zero is not defined. The function f is $x \to \frac{g(x)}{g(0)}$ and is itself not defined in the case $g(0) = 0$.

2. $f(x + y) = 1$, $f(x) = 1$, $f(y) = 1$; hence (6) becomes $1 = 1 \cdot 1$. This function also satisfies (3).

3. Equation (3) takes the form

$$\frac{Af(x+y)}{Af(x)} = \frac{Af(u+y)}{Af(u)}$$

$$\text{or } \frac{f(x)\cdot f(y)}{f(x)} = \frac{f(u)\cdot f(y)}{f(u)} \text{ since } f \text{ satisfies (6),}$$

$$\text{or } f(y) = f(y),$$

and (3) is therefore satisfied.

Answers to Exercises 4-15b. Page A-44.

1. If $f: x \to mx + b$, then
$f(x + y) = m(x + y) + b$,
$f(x)f(y) = (mx + b)(my + b) = m^2xy + bm(x + y) + b^2$.
These are the same only if $m = 0$ and $b = 0$ or 1. Hence, if $m \neq 0$, $f: x \to mx + b$ is not a solution of Equation (6).

2. If $f: x \to x^2$, then
$f(x + y) = (x + y)^2$,
$f(x)f(y) = x^2y^2$.
In particular, if $x = y = 1$, then $(x + y)^2 \neq x^2y^2$. Hence, $f(1 + 1) \neq f(1)f(1)$, and Equation (6) is not satisfied.

Answers to Exercises 4-15c. Page A-45.

1. Since $0 \cdot a = 0$ for all real numbers a, then, in particular, $0 \cdot a = 1$ is true for no real number a.
   Hence, $f(x) \neq 0$ for any x.

2. Since $f(x) \neq 0$ (from Exercise 1), we may divide Equation (8) by $f(x)$. This gives the required equation.

Answers to Exercises 4-15d. Page A-47.

1. $f(x + y) = f(x)f(y)$
   Let $y = 2x$
   $f(x + 2x) = f(x) \cdot f(2x)$.

   But we just proved that

   $f(2x) = [f(x)]^2$. Hence

   $f(3x) = [f(x)]^3$

2. To prove $f(mx) = [f(x)]^m$ for all positive integers m and real numbers x.

   a) *Initial step:* $f(1 \cdot x) = [f(x)]^1$ is obviously true.

   b) *Sequential step:* If $f(kx) = [f(x)]^k$ for any positive integer k, then

$$\begin{aligned} f[(k + 1)x] &= f(kx + x) \\ &= f(kx)f(x) && \text{by Equation (6)} \\ &= [f(x)]^k f(x) && \text{by the induction hypothesis} \\ &= [f(x)]^{k + 1} \end{aligned}$$

q.e.d.

Answers to Exercises 4-15e. Page A-49.

1. $f(1/3) = a^{1/3}$ $\quad f(1/5) = a^{1/5}$ $\quad f(4/5) = a^{4/5}$
   $f(1/4) = a^{1/4}$ $\quad f(2/5) = a^{2/5}$
   $f(3/4) = a^{3/4}$ $\quad f(3/5) = a^{3/5}$

2. $f\left(\frac{371}{1000}\right) = a^{371/1000}$

3. $f(r) = f(r\cdot 1) = [f(1)]^r = 1^r = 1$ by (13).

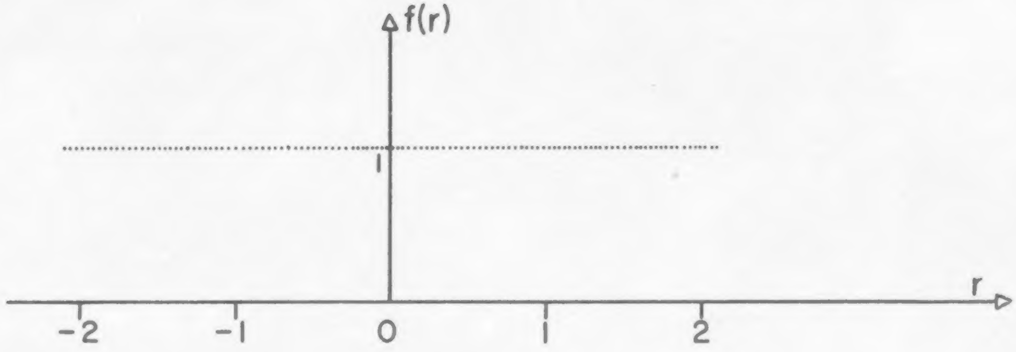
4. If a and r are both positive rational numbers, then so is $\frac{r}{a}$, and we have

$$f(r) = f(\tfrac{r}{a}\cdot a) = [f(a)]^{r/a} = 1^{r/a} = 1,$$

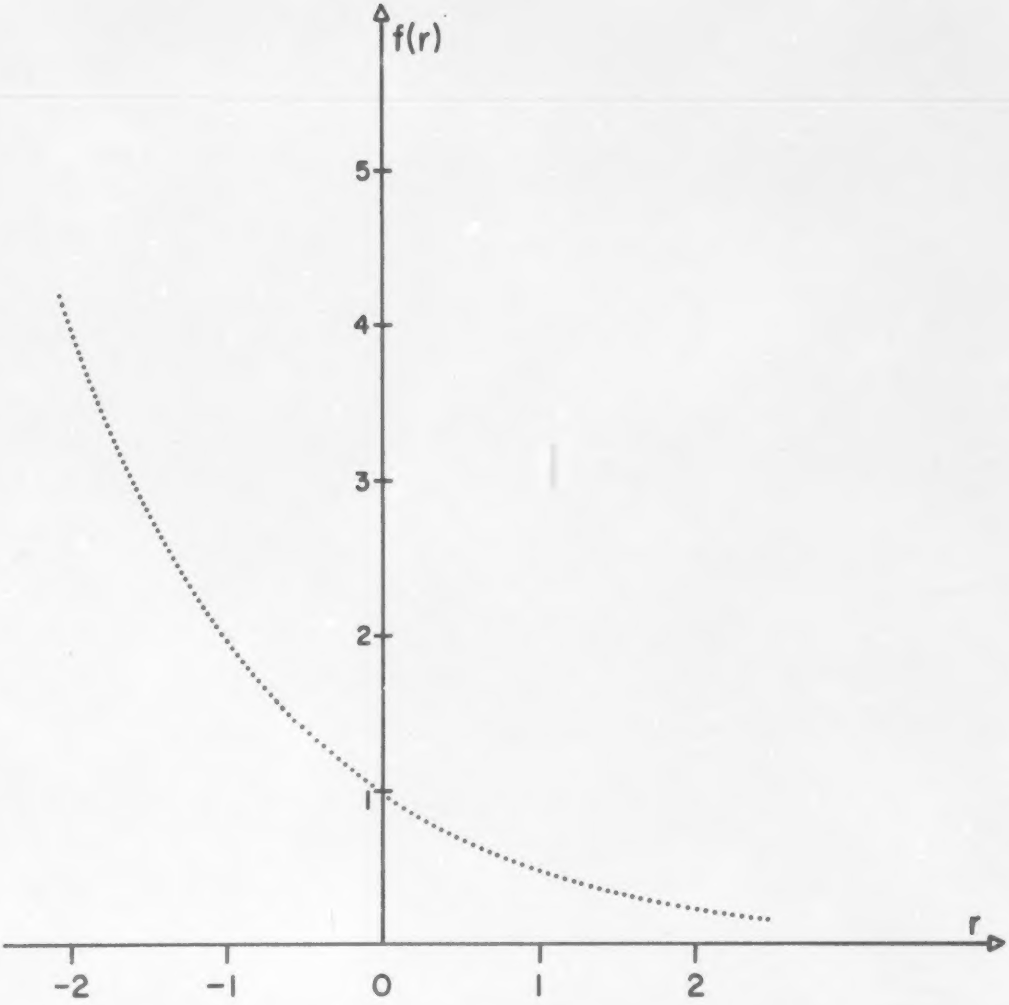
using (13).

Answers to Exercises 4-15f. Page A-51.

1. $f(r) = 1^r$



2. $f(r) = (\frac{1}{2})^r$



Answers to Exercises 4-15g. Page A-51.

1. f is the function $\log_a$: $\log_a(xy) = \log_a x + \log_a y$.

2. a) $f(0 + 1) = f(0) + f(1)$,

   but $0 + 1 = 1$,

   so $f(1) = f(0) + f(1)$

   and since f is defined for all real numbers, $f(1)$ is a number and can therefore be subtracted. This gives

   $$0 = f(0).$$

   b) $f(x-x) = f(x) + f(-x)$

   $f(0) = f(x) + f(-x)$

   $0 = f(x) + f(-x)$

   Hence

   $f(-x) = -f(x)$

   c) $f(x + x) = f(x) + f(x)$

   $f(2x) = 2f(x)$

   Similarly, $f(3x) = 3f(x)$, $f(4x) = 4f(x)$, and so on.

   Proof that $f(mx) = mf(x)$.

   Initial Step: $f(1\cdot x) = f(x) = 1\cdot f(x)$

   Sequential Step: Suppose $f(kx) = kf(x)$ for some natural number k. Then

   $f[(k + 1)x] = f(kx + x) = f(kx) + f(x)$ by (A)

   $= kf(x) + f(x)$ by the induction hypothesis

   $= (k + 1) f(x)$.

   q.e.d.

   d) $f(m\cdot\frac{x}{m}) = m\cdot f(\frac{x}{m})$

   $f(x) = mf(\frac{x}{m})$

   Hence $f(\frac{x}{m}) = \frac{1}{m}\cdot f(x)$

e) Write n for m in (E): $f(\frac{x}{n}) = \frac{1}{n} f(x)$

Now put mx for x: $f(\frac{mx}{n}) = \frac{1}{n} f(mx)$

$$= \frac{1}{n} \cdot mf(x) \quad \text{by (D)}$$

or $f(\frac{m}{n} \cdot x) = \frac{m}{n} f(x)$ for any natural numbers m and n.

Any positive rational number can be written in the form $\frac{m}{n}$, where m and n are natural numbers. Hence we have

$$f(rx) = rf(x)$$

f) If $r = 0$, we have

$$f(r \cdot x) = f(0 \cdot x) = f(0) = 0 = 0 \cdot f(x) = r \cdot f(x) \quad \text{by (B).}$$

If $r < 0$, then $-r > 0$ and

$$\begin{aligned} f(rx) &= -f(-rx) && \text{by (C)} \\ &= -(-r)f(x) && \text{by (F)} \\ &= rf(x) \end{aligned}$$

g) $f(r) = f(r \cdot 1) = rf(1)$

for all rational r by parts (e) and (f). But

$f(1) = a,$

hence $f(r) = ar.$

*h) If x is any irrational number, there are rational numbers r and s such that

$$r < x < s$$

and such that x-r and s-x are arbitrarily close to zero.

We have,

$$f(r) = ar, \quad f(s) = as.$$

If f is increasing, $a > 0$, and

$$f(r) = ar < ax < as = f(s).$$

The differences $ax - f(r) = ax - ar = a(x-r)$

and $f(s) - ax = as - ax = a(s-x)$

can be made arbitrarily close to zero. If f is defined for all real numbers x and is increasing, the only value f(x) can have is ax.

The argument when f is decreasing, and hence $a < 0$, is similar. The required conclusion follows, $f(x) = ax$, $a \neq 0$, therefore holds.

3. $f: x \rightarrow ax + b$

Answers to Exercises 4-16. Page A-57.

1. $e^{0.01} \approx 1 + 0.01 + 0.00005 \approx 1.01005$

Three terms were used. (The fourth term is $0.0000001666\cdots$)

2. $g_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + a_4x^4$

$$g_4'(x) = 1 + x + \frac{x^2}{2!} + 4a_4x^3$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = g_3(x)$$

Hence, $a_4 = \frac{1}{4 \cdot 3!} = \frac{1}{4!}$

$$g_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + a_5x^5$$

$$g_5'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + 5a_5x^4$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} = g_4(x)$$

Hence, $a_5 = \frac{1}{5 \cdot 4!} = \frac{1}{5!}$

3. $h_3(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{cx^n}{n!}$

$$h_3'(x) = 1 + x + \cdots + \frac{x^{n-2}}{(n-2)!} + \frac{cx^{n-1}}{(n-1)!}$$

$$h_3'(x) > h_3(x)$$

if $\quad \frac{cx^{n-1}}{(n-1)!} > \frac{x^{n-1}}{(n-1)!} + \frac{cx^n}{n!}$

or $\quad \frac{(c-1)x^{n-1}}{(n-1)!} > \frac{cx^n}{n!}$

This simplifies to $x < n(\frac{c-1}{c})$.

4. $e > 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} \approx 2.7167$

$e < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{2}{5!} \approx 2.7250$

Answers to Exercises 4-17. Page A-61.

1. $e^{0.2} = (e^{0.1})^2 \approx (1.105)^2 \approx 1.2210$

$e^{0.4} = (e^{0.2})^2 \approx (1.221)^2 \approx 1.4908$

$e^{0.8} = (e^{0.4})^2 \approx (1.4908)^2 \approx 2.2225$

$e = (e^{0.2})(e^{0.8}) \approx (1.221)(2.2225) \approx 2.714.$

2. $\log_{10}(e^{0.1})^{10} = 10 \log_{10} e^{0.1}$

$$\approx 10 \log_{10} 1.105 \approx 10\ (0.04336) = 0.4336$$

Antilog $0.4336 \approx 2.714$, so that $e \approx 2.714$

3. $e^{1.2} = e(e^{0.2}) = (2.714)(1.221) \approx 3.3138.$

$e^{1.2} = (e^{0.4})(e^{0.8}) \approx (1.4908)(2.2225) \approx 3.3133.$

From Table 4-6, $e^{1.2} \approx 3.3201.$

Answers to Exercises 4-18. Page A-64.

1. $\ln(1.2) = \ln(1 + .2) = 0.2 - \frac{(.2)^2}{2} + \frac{(.2)^3}{3} - \frac{(.2)^4}{4} + E$

$$= 0.1822666 \ldots + E$$

$$\approx 0.182267 + E.$$

Error $= E < \frac{(.2)^5}{5} = 0.000064$. Hence $\ln 1.2 \approx 0.1823.$

2. Using (1) to calculate $\ln 2$, the error $E < \frac{(2 - 1)^n}{n} = \frac{1}{n}$. Therefore if $E < 0.001$, we must take $\frac{1}{n} \leq 0.001$, so that $n \geq 1000$.

If n is fixed and we increase u $(u > 0)$, the error $E = \frac{u^{n+1}}{n+1}$ also increases. So long as $0 \leq u \leq 1$, we can offset this by increasing n. As u gets close to 1, however, this gets increasingly difficult, and, if $u > 1$, E gets completely out of control and actually increases with increasing n. It is therefore not surprising that (1) is not efficient for $u = 1$, that is, at the very threshold of a region in which it does not work at all. The graphs in Figure 4-12b illustrate this situation.

3. $\ln 0.8 = -\left[0.2 + \frac{(.2)^2}{2} + \frac{(.2)^3}{3} + \frac{(.2)^4}{4}\right] + E \approx -0.22307.$

4. $\ln 0.72 = \ln 0.8 + \ln 0.9 \approx -0.22307 - 0.10536 \approx -0.32843.$

5. $\ln 1.44 = \ln (1.2)^2 = 2 \ln 1.2 \approx 2(0.182267) \approx 0.36453.$

6. $\ln 2 = \ln \frac{1.44}{0.72} = \ln 1.44 - \ln 0.72 \approx 0.69296.$

7. The device used is to write

$$\ln x = -\ln(\tfrac{1}{x}) = -\ln(1 + u).$$

Now if $0 < x < .5$, then

$$\frac{1}{x} > 2 \quad \text{and} \quad u > 1.$$

But we know that the approximation (1) for $\ln(1 + u)$ cannot be used when $u > 1$.

To find $\ln 0.25$ we may proceed as follows:

$$\ln 0.25 = \ln(1/4) = -\ln 4 = -2 \ln 2.$$

From Exercise 6, $\ln 2 \approx 0.69296$.

Hence, $\ln 0.25 \approx -2(0.69296) \approx -1.38592$.

Answers to Exercises 5-15a. Page A-69.

1. 169 feet.
2. 44°.
3. Angle of rails inclined to horizontal is approximately 1.7°. Rise is approximately 270 feet.
4. 12.9 inches.
5. 13.1 inches.
6. 13,700 feet.

Answers to Exercises 5-15b. Page A-72.

1. a) $a = 8.9$

   b) $\alpha = 120^\circ$

   c) $\alpha = 11^{-\circ}$, $\beta = 115^{-\circ}$, $\gamma = 55^\circ$
2. Largest angle is opposite 12 side and $= 117^\circ$.
3. a) $d^2 = a^2 + b^2 - 2ab \cos \theta$

   b) $d^2 = a^2 + b^2 - 2ab \cos (180^\circ - \theta) = a^2 + b^2 + 2ab \cos \theta$

   c) Area of ▱ $= ab \sin \theta$.
4. 4.8" and 13.3".
5. Approximately 115 miles.
6. a) If $\gamma = 180^\circ$, then $c^2 = a^2 + b^2 - 2ab(-1)$

   $c^2 = a^2 + 2ab + b^2$

   $c^2 = (a + b)^2$

   $c = a + b$ (No triangle exists; it is a straight line.)

7. a) $\frac{1 + \cos\alpha}{2} = \frac{(b + c + a)(b + c - a)}{4bc}$

$$4bc + 4bc\cos\alpha = 2(b^2 + 2bc + c^2 - a^2)$$

$$4bc\cos\alpha = 2b^2 + 2c^2 - 2a^2$$

$$2bc\cos\alpha = b^2 + c^2 - a^2$$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \text{(Law of Cosines).}$$

Since these steps are reversible, the desired conclusion follows from the Law of Cosines.

b) $\frac{1 - \cos\alpha}{2} = \frac{(a + b - c)(a - b + c)}{4bc}$

$$4bc - 4bc\cos\alpha = 2(a^2 - b^2 + 2bc - c^2)$$

$$-4bc\cos\alpha = 2a^2 - 2b^2 - 2c^2$$

$$-2bc\cos\alpha = a^2 - b^2 - c^2$$

$$b^2 + c^2 - 2bc\cos\alpha = a^2 \quad \text{(Law of Cosines).}$$

Since these steps are reversible, the desired conclusion follows from the Law of Cosines.

Answers to Exercises 5-15c. Page A-77.

1. a) $\alpha = 82^\circ$; $a = 43.6$

   b) $\beta = 75^\circ$; $a = 14.6$; $c = 17.9$

   c) No solution

   d) No solution

   e) $\beta = 75^\circ$

   f) $\beta = 47^\circ$

   g) $\beta = 61^\circ 30'$ (Interpolate)

   h) $c = 90.8$; $\alpha = 31^\circ$.

2. $\sin \alpha = \frac{a \sin \beta}{b}$

$$\frac{\sin \alpha - \sin \beta}{\sin \alpha + \sin \beta} = \frac{\frac{a \sin \beta}{b} - \sin \beta}{\frac{a \sin \beta}{b} + \sin \beta} = \frac{\frac{a}{b} - 1}{\frac{a}{b} + 1} = \frac{a - b}{a + b} .$$

Answers to Exercises 5-16. Page A-78.

Exercises 1 to 25 are proofs that, in our opinion, require no comment.

26. $\frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi$

27. $\frac{\pi}{6} + n\pi, \frac{5\pi}{6} + n\pi$

28. $\frac{\pi}{6} + n\pi, \frac{5\pi}{6} + n\pi$

29. $n\pi$

30. $\frac{\pi}{2} + n\pi, \frac{\pi}{6} + 2n\pi, - \frac{\pi}{6} + 2n\pi$

31. $n\pi, \frac{\pi}{6} + n\pi, - \frac{\pi}{6} + n\pi$

32. $\frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi$

33. $n\pi, \frac{2\pi}{3} + 2n\pi, \frac{4\pi}{3} + 2n\pi$

34. $\frac{3\pi}{4} + n\pi$

35. $2n\pi, \frac{\pi}{3} + 2n\pi, - \frac{\pi}{3} + 2n\pi$

36. $\frac{\pi}{4} + \frac{n\pi}{2}$

37. $\frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi, \frac{3\pi}{2} + 2n\pi$

38. $\frac{\pi}{4} + n\pi, \frac{3\pi}{4} + n\pi$

39. $(2n + 1)\pi, \frac{\pi}{3} + 2n\pi, - \frac{\pi}{3} + 2n\pi$

40. $\frac{\pi}{6} + n\pi, - \frac{\pi}{6} + n\pi$

41. $\frac{\pi}{8} + \frac{n\pi}{2}$

42. $\frac{\pi}{6} + n\pi, \frac{5\pi}{6} + n\pi$

43. $2n\pi, \frac{2\pi}{3} + 2n\pi, \frac{4\pi}{3} + 2n\pi$

44. $(2n + 1)\pi, \frac{2\pi}{3} + 2n\pi, \frac{4\pi}{3} + 2n\pi$

45. $2n\pi$

46. $\frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi, \frac{3\pi}{2} + 2n\pi$

47. $2n\pi$

48. $\frac{5\pi}{6} + n\pi$

49. An exact solution is 0. Approximate solutions are ± 4.49 (near $\pm \frac{3\pi}{2}$), ± 7.72 (near $\pm \frac{5\pi}{2}$), and so forth.

50. $-\frac{\pi}{2}, 0, \frac{\pi}{2}$

Answers to Exercises 5-17. Page A-85.

1. $$\sin 0.1 = 0.1 - \frac{.001}{6} + \frac{.00001}{120} - \dots$$
$$\approx 0.1 - 0.00017 + 0.00000008$$
$$\approx 0.0998$$

From Table I, $\sin 0.1 \approx 0.0998$.

2. a) Since $x > \sin x > x - \frac{x^3}{3!}$, the error $E < \frac{x^3}{3!}$

For $E < 0.01$, $x^3 < 0.06$, and $x < 0.39$.
Hence, the error is less than 0.01 if $|x| < 0.39$.

b) Since $1 > \cos x > \frac{x^2}{2!}$, the error $E < \frac{x^2}{2!}$.

For $E < 0.01$, $x^2 < 0.02$, and $x < 0.14$.
Hence, the error is less than 0.01 if $|x| < 0.14$.

3. $$\tan x = \frac{\sin x}{\cos x} \approx \frac{x - \frac{x^3}{3} + \frac{x^5}{120}}{1 - \frac{x^2}{2} + \frac{x^4}{24}}$$

$$\approx x + \frac{x^3}{3} + \frac{2x^5}{15} .$$

Because only odd powers of x appear in this approximation, the relationship $\tan(-x) = -\tan(x)$ is suggested.

4. $e^{ix} = \cos x + i \sin x$

a) $e^{\pi i/2} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = 0 + i = i$

b) $e^{\pi i} = \cos \pi + i \sin \pi = -1 + 0 \cdot i = -1$

c) $e^{3\pi i} = \cos 3\pi + i \sin 3\pi = -1 + 0 \cdot i = -1$

d) $e^{\pi i/3} = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = \frac{1}{2} + \frac{i\sqrt{3}}{2}$

e) $e^{0.5i} = \cos 0.5 + i \sin 0.5 = 0.8776 + 0.4794\, i$